VI. LAGRANGE'S *Ballistic Problem.*

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[PLATES 1 AND 2.]

CONTENTS.

PART I.

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PART I.

1. *Introduction.*—The first part of this paper, written by LOVE, contains a theoretical solution of the problem of rational hydrodynamics which has been named by writers on ballistics, "LAGRANGE's problem"; the second part, written by PIDDUCK, gives the application to ballistics.

In the problem* it is supposed that a given mass of gas, which is initially in a uniform state, is contained in a segment of a tube of uniform section. At one end the segment of the tube is bounded by a fixed transverse section, and at the other end the tube is closed by a piston of given mass, which is initially at rest and is free to move along the tube without resistance. It is required to find the subsequent states of the gas and the motion of the piston.

Under the pressure exerted by the gas the piston begins to move, and wave-motion of finite amplitude is set up in the gas. The waves are plane. The theory of plane waves of expansion of finite amplitude has been the subject of much study,† chiefly in connection with the question of the initiation and maintenance of surfaces of discontinuity. The difficulties associated with this question do not arise in Lagrange's problem, because the waves that are generated are always waves of rarefaction, and there is no tendency to discontinuity in waves of this type. Among the results that have been obtained in the theory of plane waves of finite amplitude, two are specially important for our present purpose. The first of these is that there exist waves of the type known as "progressive waves," and that they are the only ones that can advance without discontinuity into gas at rest. They are sometimes described as "motions compatible with rest."‡ The second important result is that the equations governing the propagation of waves which are not compatible with rest can be integrated.§ Such waves will be described in the sequel as "compound waves."

The most important writings in which LAGRANGE's problem is dealt with are the memoir of HUGONIOT cited above, H. HADAMARD's 'Leçons sur la Propagation des Ondes,' Paris, 1903, and a memoir by F. GOSSOT and R. LIOUVILLE in 'Mémorial des Poudres et Salpêtres,' vol. 17, 1914, p. 1.

The problem is not rendered essentially more difficult if it is supposed that the segment of the tube occupied by gas is bounded by two movable pistons of given masses. Provision can be made for the case of a fixed end by taking the masses of the two pistons to be equal, for then there is never any velocity at the section midway between them.

The tube will be thought of as running from left to right. When the pistons begin to move progressive waves set out, one from the left-hand piston with a front proceeding towards the right, the other from the right-hand piston with a front proceeding towards the left. These waves meet at the middle section, and from that section there then sets out a compound wave, which has an advancing front, proceeding towards the right, and a receding front proceeding towards the left. This wave will be described as the

* S. D. POISSON, "Formules relatives au Mouvement du Boulet . . . extraites des Manuscrits de Lagrange," Paris, 'J. Éc. Pol.,' cah. 21 (1832).

† Reference may be made to LAMB's 'Hydrodynamics,' ch. 10.

‡ H. HUGONIOT, Paris, 'J. Éc. Pol.,' cah. 57 (1887) and cah. 58 (1889).

§ B. RIEMANN, 'Göttingen Abh.,' vol. 8 (1859–60); also 'Ges. math. Werke,' Leipzig, 1876, p. 145.

"first middle wave." When the advancing and receding fronts of the first middle wave reach the pistons the original progressive waves are obliterated, reflexions take place at the pistons, and new compound waves are generated at the pistons and encroach upon the first middle wave. These waves will be described as the "first reflected wave from the left" (or "from the right" as the case may be). The reflected waves meet at or near the middle section, from which there then sets out a new compound wave called the "second middle wave." This wave again has two fronts, one advancing and encroaching upon the first reflected wave from the right, and the other receding and encroaching upon the first reflected wave from the left. The two fronts eventually reach the pistons, and then the second middle wave will have obliterated the first reflected waves, and will itself be reflected so as to give rise to new compound waves setting out from the pistons. These will be called the "second reflected wave from the left" (or "from the right" as the case may be). The motion goes on in this way until a piston reaches an end of the tube if the tube is of finite length.

In what follows Articles 2–9 are devoted to giving such an account of the theory of plane waves of finite amplitude as seems to be necessary for the discussion of the problem. Although so much has been written about the subject, it appears to be impossible to find what is wanted in a suitable form. Articles 10, 11 contain the formulæ relating to the two progressive waves. These are already known from the work of GOSSOT and LIOUVILLE, but it seemed to be desirable, for the sake of completeness, to obtain them anew. Articles 12–17 deal with the first middle wave. Sufficient indications of the method of determining this wave have been given by the same writers for the case of equal pistons. The really formidable difficulties of the problem begin to present themselves when an attempt is made to discuss the waves reflected from the moving pistons. In Articles 18–25 an approximate method of solution is found. It seems to be capable of giving results for the first reflected waves correct to any desired order of accuracy. In Articles 26–31 the second middle wave is determined. However far the approximation to the first reflected waves is carried, the second middle wave answering to them can be found by the method here given. Articles 32–40 are devoted to the determination of the second reflected waves. The method used for the first reflected waves does not give a sufficiently close approximation, and a new method is applied. Numerical calculation of a particular example showed that all information that can be of practical importance may be obtained from a solution which does not go beyond the determination of these waves. The results of this calculation belong properly to the second part of the paper.

THEORY OF PLANE WAVES OF FINITE AMPLITUDE.

2. *General Equations.*—The motion is supposed to take place in an unlimited straight tube of uniform cross-section ω. Let x be a co-ordinate measured along the tube, and specifying the position at time t of a plane of particles, which, when $t = 0$, is in

the position specified by x_0. At the time $t = 0$ the gas is supposed to be at rest. Let p_0 and ρ_0 denote the undisturbed pressure and density, supposed uniform, and let p, ρ, u denote the pressure, density and velocity at time t for the particles specified by x_0. The equation of continuity is

$$\rho \frac{\partial x}{\partial x_0} = \rho_0,$$

and the equation of motion is

$$\rho \frac{\partial^2 x}{\partial t^2} \frac{\partial x}{\partial x_0} = -\frac{\partial p}{\partial x_0}.$$

On introducing u, which is $\partial x/\partial t$, these equations become

$$\frac{\partial u}{\partial x_0} = -\frac{\rho_0}{\rho^2}\frac{\partial \rho}{\partial t}, \qquad \frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_0}.$$

It is supposed that p is a uniform function of ρ, and it is convenient to introduce, after RIEMANN, a quantity σ by the defining equation

$$d\sigma = \frac{1}{\rho}\sqrt{\left(\frac{dp}{d\rho}\right)}\, d\rho$$

and the condition that $\sigma = 0$ when $\rho = 0$. Then the equations become

$$\frac{\partial \sigma}{\partial t} + \Pi \frac{\partial u}{\partial x_0} = 0, \qquad \frac{\partial u}{\partial t} + \Pi \frac{\partial \sigma}{\partial x_0} = 0,$$

where Π is a function of ρ defined by the equation

$$\Pi = \frac{\rho}{\rho_0}\sqrt{\left(\frac{dp}{d\rho}\right)}.$$

The quantity Π, which is of the dimensions of a velocity, may be regarded as a known function of σ. The value of Π when $\rho = \rho_0$ is the velocity of sound waves of small amplitude in the undisturbed state of the gas. This will be denoted by a. The equations are of Lagrangian type, and x_0 and t are the independent variables. The quantities ρ and p, like Π, can be regarded as known functions of σ. The value of σ when $\rho = \rho_0$ will be denoted by σ_0.

3. *Progressive Waves.*—Two quantities r and s may be introduced, after RIEMANN, by the equations

$$\sigma + u = 2r, \qquad \sigma - u = 2s,$$

or

$$\sigma = r + s, \qquad u = r - s.$$

The equations of continuity and motion then give the two equations

$$\frac{\partial r}{\partial t} + \Pi \frac{\partial r}{\partial x_0} = 0, \qquad \frac{\partial s}{\partial t} - \Pi \frac{\partial s}{\partial x_0} = 0.$$

If s is constant, the second of these equations becomes an identity, and the first can be integrated in the form

$$r = \mathrm{F}(x_0 - \Pi t),$$

where F denotes an arbitrary function. This can be proved easily, and the equation can be written

$$x_0 - \Pi t = f(\sigma).$$

In like manner, when r is constant, the first of the two equations becomes an identity, and the second can be integrated in the form

$$x_0 + \Pi t = f(\sigma).$$

A motion with constant r or constant s is described as a "progressive wave." A wave with constant s is propagated in the direction of increase of x, with velocity Π, which depends upon the constant value of s and the local value of r. This is the velocity relative to the *medium*, not the velocity relative to the tube. Similar statements hold for a wave of constant r.

4. *Motion of a Junction.*—When a wave is transmitted into gas at rest, or into a region where there is some other state of motion, there may be discontinuity in the values of the pressure, &c., in the two regions separated by the front of the wave. We consider here the case where there is no such discontinuity, but, while the pressure, &c., have the same values on the two sides of any plane $x =$ const., the laws of variation of these quantities on the two sides of a wave-front are different. We describe such a moving wave-front as a "junction." Our immediate object is to determine the velocity of a junction relative to the medium. We shall attain this object by supposing that there are very slight differences between the values of any of the quantities on the two sides of the wave-front.

Let w denote the velocity of the junction relative to the medium. In a very short time δt a mass equal to $\rho_0 \omega w \, \delta t$ has its motion and state changed from those specified by u, p, ρ, to those specified by $u + \Delta u$, $p + \Delta p$, $\rho + \Delta \rho$. The increment of momentum must be equal to the impulse of the difference of pressure, and therefore we have the equation

$$\rho_0 \omega w \, \delta t \, \Delta u = \omega \, \Delta p \, \delta t.$$

Further, the work done during the interval δt by the external pressures on the ends of this element of mass must be equal to the sum of the increments of the kinetic and intrinsic energies of the element. Now the changes of state being adiabatic and very slight, the increment of the intrinsic energy per unit of mass may be put equal to

$$-p\Delta(1/\rho),$$

and therefore we have the equation

$$\omega(p + \Delta p)(u + \Delta u)\,\delta t - \omega p u \, \delta t = \tfrac{1}{2}\rho_0 \omega w \, \delta t \, \{(u + \Delta u)^2 - u^2\} - \rho_0 \omega w \, \delta t p \, \Delta(1/\rho).$$

The two equations containing Δp and Δu give

$$\Delta p = \rho_0 w\, \Delta u$$

and

$$p\, \Delta u + u\, \Delta p = \rho_0 w u\, \Delta u + \rho_0 w\, (p/\rho^2)\, \Delta \rho.$$

The terms $u\Delta p$ and $\rho_0 w u \Delta u$ in the second of these equations cancel, and then, by eliminating Δu between the two equations, we find

$$w^2 = \frac{\rho^2}{\rho_0^{\,2}} \frac{\Delta p}{\Delta \rho}.$$

Since there is no actual discontinuity and p is a uniform function of ρ, we may replace $\Delta p/\Delta\rho$ by $dp/d\rho$, and thus obtain the equation

$$w^2 = \frac{\rho^2}{\rho_0^{\,2}} \frac{dp}{d\rho},$$

which shows that the velocity of the junction relative to the medium is that which was previously denoted by Π.

If motion is set up in one part of the gas, and advances into previously undisturbed gas, the value of ρ at the junction is ρ_0, and therefore the velocity of the front of the wave, relative to the medium or to the tube, is that which has been denoted by a.

5. *Nature of the Motion in a Compound Wave.*—Important results can be obtained by regarding x_0 and t as functions of r and s. On interchanging the dependent and independent variables in the equations

$$\frac{\partial r}{\partial t} + \Pi \frac{\partial r}{\partial x_0} = 0, \qquad \frac{\partial s}{\partial t} - \Pi \frac{\partial s}{\partial x_0} = 0,$$

we obtain the equations

$$\frac{\partial x_0}{\partial s} - \Pi \frac{\partial t}{\partial s} = 0, \qquad \frac{\partial x_0}{\partial r} + \Pi \frac{\partial t}{\partial r} = 0.$$

Now the differentials of x_0 and t are always connected with those of r and s by the formulæ

$$dx_0 = \frac{\partial x_0}{\partial r} dr + \frac{\partial x_0}{\partial s} ds, \qquad dt = \frac{\partial t}{\partial r} dr + \frac{\partial t}{\partial s} ds.$$

Hence the places in the medium, and the times, at which any particular value of r is found, vary according to the formulæ

$$dx_0 = \frac{\partial x_0}{\partial s} ds = \Pi \frac{\partial t}{\partial s} ds, \qquad dt = \frac{\partial t}{\partial s} ds = \frac{dx_0}{\Pi},$$

and thus it appears that any value of r is transmitted through the medium, in the direction of increase of x, with the velocity Π. In like manner it can be shown that any value of s is transmitted in the opposite direction with the same local velocity.

We have seen that the velocity of a junction relative to the medium is the value of Π at the junction, and it follows that the value of r remains constant along a junction which travels in the direction of increase of x. If the junction travels in the opposite direction, the value of s at the junction remains constant.

The motion consequent upon any initial conditions consists in the transfer of the existing values of r and s through the medium with the variable velocity already described. New values of r and s can be generated at boundaries and transferred through the medium.

6. *General Analysis of Compound Waves.*—When the dependent and independent variables are interchanged in the equations

$$\frac{\partial \sigma}{\partial t} + \Pi \frac{\partial u}{\partial x_0} = 0, \qquad \frac{\partial u}{\partial t} + \Pi \frac{\partial \sigma}{\partial x_0} = 0,$$

there result the equations

$$\frac{\partial x_0}{\partial u} + \Pi \frac{\partial t}{\partial \sigma} = 0, \qquad \frac{\partial x_0}{\partial \sigma} + \Pi \frac{\partial t}{\partial u} = 0.$$

The first of these shows that there exists a function Z of σ and u which has the properties expressed by the equations

$$x_0 = -\Pi \frac{\partial Z}{\partial \sigma}, \qquad t = \frac{\partial Z}{\partial u},$$

and then the second shows that Z satisfies the differential equation

$$\frac{\partial}{\partial \sigma}\left(\Pi \frac{\partial Z}{\partial \sigma}\right) - \Pi \frac{\partial^2 Z}{\partial u^2} = 0.$$

If Z can be found in accordance with this equation, the values of x_0 and t answering to any simultaneous value of σ and u can be deduced.

There is a relation between Z and x, which can be obtained very simply by introducing for a moment a quantity ϖ by the equation

$$\varpi = \rho_0/\rho,$$

for then we have

$$-\frac{d\sigma}{\Pi} = -\frac{\rho_0}{\rho^2}\, d\rho = d\varpi,$$

and it follows that we have at once

$$x_0 = \frac{\partial Z}{\partial \varpi}, \qquad t = \frac{\partial Z}{\partial u},$$

and

$$\varpi = \frac{\partial x}{\partial x_0}, \qquad u = \frac{\partial x}{\partial t}.$$

These are the relations of duality familiar in discussions of partial differential equations,* and we may put

$$Z = x_0 \frac{\partial x}{\partial x_0} + t \frac{\partial x}{\partial t} - x.$$

Actually Z could differ from the right-hand member of this equation by a constant, but as such a constant would be irrelevant, the above will be taken as the relation between Z and x.

The equation satisfied by Z can be written in either of the forms

$$\frac{\partial^2 Z}{\partial \sigma^2} + \left(\frac{1}{\Pi}\frac{d\Pi}{d\sigma}\right)\frac{\partial Z}{\partial \sigma} - \frac{\partial^2 Z}{\partial u^2} = 0$$

or

$$\frac{\partial^2 Z}{\partial r\, \partial s} + \left(\frac{1}{2\Pi}\frac{d\Pi}{d\sigma}\right)\left(\frac{\partial Z}{\partial r} + \frac{\partial Z}{\partial s}\right) = 0.$$

7. *Relation between Pressure and Density.*—The analysis of the problem is not rendered more difficult if the adiabatic relation between pressure p and volume v is taken in the form $p\,(v - b)^\gamma = \text{const.}$ instead of the more ordinary form $pv^\gamma = \text{const.}$, and the former is more suitable for the applications which we have in view. We shall accordingly take the relation between pressure and density to be

$$p\left(\frac{1}{\rho} - \frac{1}{\beta}\right)^\gamma = p_0\left(\frac{1}{\rho_0} - \frac{1}{\beta}\right)^\gamma,$$

where β and γ are constants. Then the following results can be obtained without difficulty :—

$$\sigma = \frac{2}{\gamma - 1}\left\{\frac{p_0\gamma\,(\beta - \rho_0)}{\beta\rho_0}\right\}^{\frac{1}{2}}\left(\frac{\beta - \rho_0}{\rho_0}\,\frac{\rho}{\beta - \rho}\right)^{(\gamma-1)/2},$$

$$\sigma_0 = \frac{2}{\gamma - 1}\left\{\frac{p_0\gamma\,(\beta - \rho_0)}{\beta\rho_0}\right\}^{\frac{1}{2}},$$

$$a = \left\{\frac{p_0\gamma\beta}{\rho_0\,(\beta - \rho_0)}\right\}^{\frac{1}{2}},$$

$$p = p_0\,(\sigma/\sigma_0)^{2n+1},$$

$$\Pi = a\,(\sigma/\sigma_0)^{2n},$$

where $2n$ has been written for $(\gamma + 1)/(\gamma - 1)$.

The equation for Z can now be written

$$\frac{\partial^2 Z}{\partial \sigma^2} + \frac{2n}{\sigma}\frac{\partial Z}{\partial \sigma} - \frac{\partial^2 Z}{\partial u^2} = 0,$$

or

$$\frac{\partial^2 Z}{\partial r\, \partial s} + \frac{n}{r + s}\left(\frac{\partial Z}{\partial r} + \frac{\partial Z}{\partial s}\right) = 0.$$

* The reduction of the equations governing the propagation of plane waves of finite amplitude to a single partial differential equation of the second order was effected by RIEMANN, who worked with "Eulerian" equations. The use of the principle of duality to connect Z and x was noted by HADAMARD.

8. *Integration of the Equation in Special Cases.*—When n is a positive integer, the equation can be integrated. We write for a moment D for $\partial/\partial u$, and observe that, if D were a constant, the equation

$$\frac{\partial^2 Z}{\partial\sigma^2}+\frac{2n}{\sigma}\frac{\partial Z}{\partial\sigma}-D^2 Z=0$$

would be a form of RICCATI's equation, and could be integrated in the form

$$Z=\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^{n-1}\frac{(e^{\sigma D}A+e^{-\sigma D}B)}{\sigma},$$

where A and B are independent of σ. Treating them as functions of u, we obtain the general primitive of the equation for Z in the form

$$Z=\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^{n-1}\left\{\frac{F(\sigma+u)+f(\sigma-u)}{\sigma}\right\}.$$

9. *More General Integration.*—Interpreting the variables r and s as the co-ordinates of a point in a plane, RIEMANN showed how to integrate the equation for Z when the values of this function and its first differential coefficients are given along an arc of a curve in the plane. If V satisfies the "adjoint" equation

$$\frac{\partial^2 V}{\partial r\, ds}-n\left(\frac{\partial}{\partial r}+\frac{\partial}{\partial s}\right)\left(\frac{V}{r+s}\right)=0,$$

the integral

$$\iint\left[\frac{\partial}{\partial r}\left\{V\left(\frac{\partial Z}{\partial s}+\frac{nZ}{r+s}\right)\right\}-\frac{\partial}{\partial s}\left\{Z\left(\frac{\partial V}{\partial r}-\frac{nV}{r+s}\right)\right\}\right]dr\,ds,$$

taken over any area in the plane, is equal to

$$\iint\left[V\left\{\frac{\partial^2 Z}{\partial r\,\partial s}+\frac{n}{r+s}\left(\frac{\partial Z}{\partial r}+\frac{\partial Z}{\partial s}\right)\right\}-Z\left\{\frac{\partial^2 V}{\partial r\,\partial s}-n\left(\frac{\partial}{\partial r}+\frac{\partial}{\partial s}\right)\left(\frac{V}{r+s}\right)\right\}\right]dr\,ds,$$

and therefore vanishes. It follows that the line-integral

$$\int V\left(\frac{\partial Z}{\partial s}+\frac{nZ}{r+s}\right)ds+Z\left(\frac{\partial V}{\partial r}-\frac{nV}{r+s}\right)dr$$

taken round the boundary of the area vanishes.

Let the values of Z and its first differential coefficients be given along an arc AC of a curve, and let P be a point which is not on the arc. Through P let lines PA and PC be drawn parallel to the axes of r and s, and let the area of integration be that bounded

by the arc AC and the lines CP and PA. The contribution of CP to the line-integral is

$$\int_{\mathrm{CP}} \mathrm{V}\left(\frac{\partial \mathrm{Z}}{\partial s}+\frac{n\mathrm{Z}}{r+s}\right) ds,$$

which may be written

$$[\mathrm{VZ}]_{\mathrm{P}}-[\mathrm{VZ}]_{\mathrm{C}}-\int_{\mathrm{CP}} \mathrm{Z}\left(\frac{\partial \mathrm{V}}{\partial s}-\frac{n\mathrm{V}}{r+s}\right) ds.$$

The contribution of PA to the line-integral is

$$\int_{\mathrm{PA}} \mathrm{Z}\left(\frac{\partial \mathrm{V}}{\partial r}-\frac{n\mathrm{V}}{r+s}\right) dr.$$

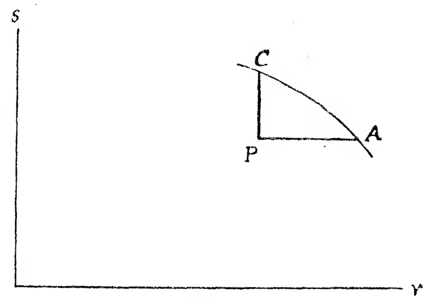


Fig. 1.

Now we can find a function to satisfy the equation for V, to make $\mathrm{V}=1$ at P, and so that, along CP, where r has the same value as at P, $\partial \mathrm{V}/\partial s = n\mathrm{V}/(r+s)$, and along PA, where s has the same value as at P, $\partial \mathrm{V}/\partial r = n\mathrm{V}/(r+s)$. Then the value of Z at P is

$$[\mathrm{VZ}]_{\mathrm{C}}-\int_{\mathrm{AC}}\left\{\mathrm{V}\left(\frac{\partial \mathrm{Z}}{\partial s}+\frac{n\mathrm{Z}}{r+s}\right) ds+\mathrm{Z}\left(\frac{\partial \mathrm{V}}{\partial r}-\frac{n\mathrm{V}}{r+s}\right) dr.\right.$$

The required function V can be shown, after Riemann, to be given by the equation

$$\mathrm{V}=\left(\frac{r+s}{r'+s'}\right)^{n} \mathrm{F}(n,\ 1-n,\ 1,\ \xi),$$

where F is the symbol for the hypergeometric series,

$$\xi=-\frac{(r-r')(s-s')}{(r+s)(r'+s')},$$

and r', s' are the co-ordinates of P.

It may be observed that if n is an integer the series terminates, and V like Z is expressible in a finite form.

It will be useful hereafter to note the formulæ

$$\frac{\partial \mathrm{V}}{\partial r}-\frac{n\mathrm{V}}{r+s}=-\frac{(r+s)^{n-2}(s+r')(s-s')}{(r'+s')^{n+1}} \frac{d}{d\xi} \mathrm{F}(n,\ 1-n,\ 1,\ \xi),$$

$$\frac{\partial \mathrm{V}}{\partial s}-\frac{n\mathrm{V}}{r+s}=-\frac{(r+s)^{n-2}(r+s')(r-r')}{(r'+s')^{n+1}} \frac{d}{d\xi} \mathrm{F}(n,\ 1-n,\ 1,\ \xi).$$

The Progressive Waves in Lagrange's Problem.

10. *The Progressive Wave from the Left.*—Let the positive sense of the axis of x be from left to right, and let the initial positions of the two pistons be given by $x_0 = 0$ and $x_0 = c$, where c is positive. We shall denote the mass of the piston at $x_0 = 0$ by M, and that of the other piston by m.

The progressive wave generated at the piston M is determined by the equation of motion of this piston. This equation is

$$\mathrm{M}\frac{\partial u}{\partial t} = -\omega p,$$

and it must hold at $x_0 = 0$ for all positive values of t. It may be written

$$\mathrm{M}\frac{\partial u}{\partial t} = -\omega p_0\left(\frac{\sigma}{\sigma_0}\right)^{2n+1},$$

and, since in the progressive wave s is constant and equal to $\frac{1}{2}\sigma_0$, or $\sigma - u = \sigma_0$, it gives

$$dt = -\frac{\mathrm{M}}{\omega p_0}\left(\frac{\sigma_0}{\sigma}\right)^{2n+1} d\sigma,$$

from which, since $\sigma = \sigma_0$ when $t = 0$, we have at $x_0 = 0$

$$t = \frac{\mathrm{M}\sigma_0}{2n\omega p_0}\left\{\left(\frac{\sigma_0}{\sigma}\right)^{2n} - 1\right\}.$$

Put for brevity

$$\mathrm{H} = \mathrm{M}\sigma_0 a/2n\omega p_0,$$

then we have the values of σ and t at $x_0 = 0$ connected by the equation

$$t = \frac{\mathrm{H}}{a}\left\{\left(\frac{\sigma_0}{\sigma}\right)^{2n} - 1\right\}.$$

Now in the progressive wave we have

$$x_0 - \mathrm{H}t = f(\sigma),$$

where the function f is to be found from the condition that at $x_0 = 0$ the above relation holds between σ and t. Hence we find

$$f(\sigma) = -\mathrm{H}\left\{1 - \left(\frac{\sigma}{\sigma_0}\right)^{2n}\right\},$$

and the progressive wave formula can be written

$$\frac{at + \mathrm{H}}{x_0 + \mathrm{H}} = \left(\frac{\sigma_0}{\sigma}\right)^{2n}.$$

In the motion described by these formulæ any plane of particles, specified by a value of x_0 in the interval $\frac{1}{2}c > x_0 > 0$, remains at rest until $t = x_0/a$, and then moves with a velocity u, which is equal to $\sigma - \sigma_0$. Therefore the value of x answering to these particles at any subsequent time is given by the equation

$$x = x_0 + \int_{x_0/a}^{t} u\,dt = x_0 + \int_{\sigma_0}^{\sigma} (\sigma_0 - \sigma)\, 2n \frac{x_0 + \mathrm{H}}{a} \frac{\sigma_0^{2n}}{\sigma^{2n+1}}\, d\sigma,$$

or

$$x = x_0 + \frac{\sigma_0}{a}(x_0 + \mathrm{H})\left[\frac{2n}{2n-1}\left\{\left(\frac{\sigma_0}{\sigma}\right)^{2n-1} - 1\right\} - \left\{\left(\frac{\sigma_0}{\sigma}\right)^{2n} - 1\right\}\right].$$

This equation holds so long as the plane of particles is in the region occupied by the progressive wave. In particular, the displacement of the piston M is given by the equation

$$x = -\frac{2n}{2n-1}(\sigma_0 - \sigma)\frac{\mathrm{H}}{a} + \left(\frac{2n}{2n-1}\sigma - \sigma_0\right)t,$$

in which

$$t = \frac{\mathrm{H}}{a}\left\{\left(\frac{\sigma_0}{\sigma}\right)^{2n} - 1\right\}.$$

The corresponding values of Z are found from the formula

$$\mathrm{Z} = x_0 \frac{\partial x}{\partial x_0} + ut - x,$$

in which

$$\frac{\partial x}{\partial x_0} = \frac{\rho_0}{\rho}, \qquad \frac{\beta - \rho_0}{\rho_0}\frac{\rho}{\beta - \rho} = \left(\frac{\sigma}{\sigma_0}\right)^{2n-1},$$

to be

$$\frac{\mathrm{H}}{a}\left\{\frac{2n}{2n-1}\sigma_0 - \sigma - \frac{\sigma_0}{2n-1}\left(\frac{\sigma_0}{\sigma}\right)^{2n-1}\right\}.$$

11. *The Progressive Wave from the Right.*—The equation of motion of the piston m is

$$m\frac{\partial u}{\partial t} = \omega p,$$

and we put

$$h = m\sigma_0 a / 2n\omega p_0.$$

Since in the progressive wave r is constant and equal to $\frac{1}{2}\sigma_0$, or $\sigma + u = \sigma_0$, the values of σ and t at $x_0 = c$ are found to be connected by the equation

$$t = \frac{h}{a}\left\{\left(\frac{\sigma_0}{\sigma}\right)^{2n} - 1\right\},$$

and then the progressive wave formula is found to be

$$\frac{at + h}{c + h - x_0} = \left(\frac{\sigma_0}{\sigma}\right)^{2n}.$$

The value of x for any plane of particles specified by a value of x_0 in the interval $\frac{1}{2}c < x_0 < c$, and for any time later than that given by $t = (c - x_0)/a$, is found to be given by the equation

$$x = x_0 + \frac{\sigma_0}{a}(c+h-x_0)\left[\left\{\left(\frac{\sigma_0}{\sigma}\right)^{2n}-1\right\} - \frac{2n}{2n-1}\left\{\left(\frac{\sigma_0}{\sigma}\right)^{2n-1}-1\right\}\right],$$

which holds so long as the plane of particles is in the region occupied by the progressive wave. In particular, the displacement of the piston m is given by the equation

$$x = c + \frac{2n}{2n-1}(\sigma_0-\sigma)\frac{h}{a} - \left(\frac{2n}{2n-1}\sigma - \sigma_0\right)t,$$

in which

$$t = \frac{h}{a}\left\{\left(\frac{\sigma_0}{\sigma}\right)^{2n}-1\right\}.$$

The formula for Z is found to be

$$Z = \frac{c+h}{(2n-1)a}\sigma_0\left\{\left(\frac{\sigma_0}{\sigma}\right)^{2n-1}-1\right\} - \frac{h}{a}(\sigma_0-\sigma).$$

The First Middle Wave.

12. *Conditions satisfied at the Receding Front.*—In the progressive wave from the left s is constant and r variable. The greatest value of r, which is the undisturbed value $\frac{1}{2}\sigma_0$, travels at the front of the wave, and continually diminishing values of r, generated at the piston M, travel after it. Similar statements, with r and s interchanged, hold for the progressive wave from the right. The fronts of both waves travel along the tube with velocity a. When they reach the middle section, a compound wave begins to be generated there, and transmitted in both directions, encroaching upon the original progressive waves. This wave has a receding front, along which s is constant, travelling towards the left, and an advancing front, along which r is constant, travelling towards the right. The constant values of r and s at the two fronts are equal, and each of them is $\frac{1}{2}\sigma_0$.

At the receding front the variations of x_0 and t are connected by the equation

$$dx_0 + \Pi\, dt = 0,$$

while the values of x_0, t and σ are connected by the progressive wave formula, which can be written

$$x_0 - \Pi t + \mathrm{H}\left(1-\frac{\Pi}{a}\right) = 0,$$

so that the variations of x_0, t and Π are connected by the equation

$$dx_0 - (\Pi\, dt + t\, d\Pi) - \frac{\mathrm{H}}{a}d\Pi = 0.$$

On elimination of dx_0 there results the equation

$$2\Pi\, dt + \left(t + \frac{\mathrm{H}}{a}\right) d\Pi = 0,$$

which can be integrated in the form

$$\left(t + \frac{\mathrm{H}}{a}\right)^2 \Pi = \text{const.}$$

To determine the constant there is the condition that when $x_0 = \frac{1}{2}c$ and $t = \frac{1}{2}c/a$, the value of Π is a. Hence at the receding front we have

$$(at + \mathrm{H})^2\, \Pi = (\tfrac{1}{2}c + \mathrm{H})^2\, a,$$

or

$$(x_0 + \mathrm{H})\,(at + \mathrm{H}) = (\tfrac{1}{2}c + \mathrm{H})^2.$$

13. *Conditions satisfied at the Advancing Front.*—At the advancing front we have in like manner

$$dx_0 - \Pi\, dt = 0 \quad \text{and} \quad c + h - x_0 - \Pi\left(t + \frac{h}{a}\right) = 0,$$

leading to

$$(at + h)^2\, \Pi = (\tfrac{1}{2}c + h)^2\, a$$

and

$$(c + h - x_0)\,(at + h) = (\tfrac{1}{2}c + h)^2.$$

14. *Conditions determining the First Middle Wave.*—It will now be convenient to restrict the value of n to be an integer. This happens when γ has one of the values 3, 5/3, 7/5, 9/7, 11/9, . . . With a view to applications, in which the value of γ is $1 \cdot 2$ nearly, we shall take the value 11/9 for γ, or 5 for n. Then in any compound wave Z has the form

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial \sigma}\right)^4 \left\{\frac{\mathrm{F}(\sigma + u) + f(\sigma - u)}{\sigma}\right\},$$

or

$$\begin{aligned} &105\sigma^{-9}\,\mathrm{F}(\sigma+u) - 105\sigma^{-8}\,\mathrm{F}^{(1)}(\sigma+u) + 45\sigma^{-7}\,\mathrm{F}^{(2)}(\sigma+u) - 10\sigma^{-6}\,\mathrm{F}^{(3)}(\sigma+u) + \sigma^{-5}\,\mathrm{F}^{(4)}(\sigma+u) \\ &+ 105\sigma^{-9} f(\sigma-u) - 105\sigma^{-8} f^{(1)}(\sigma-u) + 45\sigma^{-7} f^{(2)}(\sigma-u) - 10\sigma^{-6} f^{(3)}(\sigma-u) + \sigma^{-5} f^{(4)}(\sigma-u), \end{aligned}$$

where $\mathrm{F}^{(1)}$, $\mathrm{F}^{(2)}$, and so on stand for the first, second, &c., differential coefficients of the function F with respect to its argument. We have to determine the unknown functions from the values of Z at the advancing and receding fronts.

At the advancing front, where $r = \frac{1}{2}\sigma_0$ and $\sigma = \frac{1}{2}\sigma_0 + s$, we have

$$\mathrm{Z} = \tfrac{1}{9}(c + h)\,\frac{\sigma_0}{a}\left\{\left(\frac{\sigma_0}{\frac{1}{2}\sigma_0 + s}\right)^9 - 1\right\} - \frac{h}{a}\,(\tfrac{1}{2}\sigma_0 - s),$$

and at the receding front, where $s = \frac{1}{2}\sigma_0$ and $\sigma = \frac{1}{2}\sigma_0 + r$, we have

$$Z = -\tfrac{1}{9}H\frac{\sigma_0}{a}\left\{\left(\frac{\sigma_0}{\frac{1}{2}\sigma_0+r}\right)^9-1\right\}+\frac{H}{a}\left(\tfrac{1}{2}\sigma_0-r\right).$$

15. *Determination of the First Middle Wave.*—To determine Z from these conditions we may have recourse to RIEMANN's method, taking the curve AC to consist of segments of two lines AB and BC, which meet at the point B, where $r = s = \frac{1}{2}\sigma_0$, and are parallel respectively to the axes of s and r.

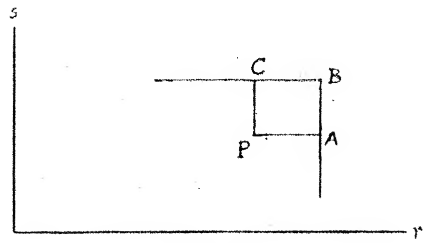


Fig. 2.

We have then

$$\int_{CP} V\left(\frac{\partial Z}{\partial s}+\frac{5Z}{r+s}\right)ds+\int_{PA} Z\left(\frac{\partial V}{\partial r}-\frac{5V}{r+s}\right)dr$$
$$+\int_{AB} V\left(\frac{\partial Z}{\partial s}+\frac{5Z}{r+s}\right)ds+\int_{BC} Z\left(\frac{\partial V}{\partial r}-\frac{5V}{r+s}\right)dr = 0.$$

This equation is

$$[VZ]_P-[VZ]_C-\int_{CP} Z\left(\frac{\partial V}{\partial s}-\frac{5V}{r+s}\right)ds+\int_{PA} Z\left(\frac{\partial V}{\partial r}-\frac{5V}{r+s}\right)dr$$
$$+[VZ]_B-[VZ]_A-\int_{AB} Z\left(\frac{\partial V}{\partial s}-\frac{5V}{r+s}\right)ds+\int_{BC} Z\left(\frac{\partial V}{\partial r}-\frac{5V}{r+s}\right)dr = 0,$$

or

$$Z(r', s') = [VZ]_A-[VZ]_B+[VZ]_C+\int_{AB} Z\left(\frac{\partial V}{\partial s}-\frac{5V}{r+s}\right)ds-\int_{BC} Z\left(\frac{\partial V}{\partial r}-\frac{5V}{r+s}\right)dr,$$

where r', s' are the co-ordinates of P. Also at A we have

$$r = \tfrac{1}{2}\sigma_0,\quad s = s',\quad \zeta = 0,\quad V = \left(\frac{\frac{1}{2}\sigma_0+s'}{r'+s'}\right)^5,\quad Z = \tfrac{1}{9}(c+h)\frac{\sigma_0}{a}\left\{\left(\frac{\sigma_0}{\frac{1}{2}\sigma_0+s'}\right)^9-1\right\}-\frac{h}{a}\left(\tfrac{1}{2}\sigma_0-s'\right),$$

so that

$$[VZ]_A = \tfrac{1}{9}(c+h)\frac{\sigma_0}{a}\left\{\frac{\sigma_0^9}{(r'+s')^5\left(\frac{1}{2}\sigma_0+s'\right)^4}-\left(\frac{\frac{1}{2}\sigma_0+s'}{r'+s'}\right)^5\right\}-\frac{h}{a}\left(\tfrac{1}{2}\sigma_0-s'\right)\left(\frac{\frac{1}{2}\sigma_0+s'}{r'+s'}\right)^5.$$

At B we have

$$r = s = \tfrac{1}{2}\sigma_0,\qquad Z = 0,$$

so that

$$[VZ]_B = 0.$$

At C we have

$$r = r',\quad s = \tfrac{1}{2}\sigma_0,\quad \zeta = 0,\quad V = \left(\frac{\frac{1}{2}\sigma_0+r'}{r'+s'}\right)^5,\quad Z = -\tfrac{1}{9}H\frac{\sigma_0}{a}\left\{\left(\frac{\sigma_0}{\frac{1}{2}\sigma_0+r'}\right)^9-1\right\}+\frac{H}{a}\left(\tfrac{1}{2}\sigma_0-r'\right),$$

so that

$$[VZ]_C = -\tfrac{1}{9}H\frac{\sigma_0}{a}\left\{\frac{\sigma_0^9}{(r'+s')^5\left(\frac{1}{2}\sigma_0+r'\right)^4}-\left(\frac{\frac{1}{2}\sigma_0+r'}{r'+s'}\right)^5\right\}+\frac{H}{a}\left(\tfrac{1}{2}\sigma_0-r'\right)\left(\frac{\frac{1}{2}\sigma_0+r'}{r'+s'}\right)^5.$$

Along AB we have

$$Z = \tfrac{1}{9}(c+h)\frac{\sigma_0}{a}\left\{\left(\frac{\sigma_0}{\frac{1}{2}\sigma_0+s}\right)^9-1\right\}-\frac{h}{a}(\tfrac{1}{2}\sigma_0-s),$$

$$\frac{\partial V}{\partial s}-\frac{5V}{r+s}=\frac{(\frac{1}{2}\sigma_0+s')(\frac{1}{2}\sigma_0-r')(\frac{1}{2}\sigma_0+s)^3}{(r'+s')^6}(20-180\zeta+420\zeta^2-280\zeta^3),$$

where

$$\zeta=-\frac{(\frac{1}{2}\sigma_0-r')(s-s')}{(r'+s')(\frac{1}{2}\sigma_0+s)},$$

so that

$$\int_{AB}Z\left(\frac{\partial V}{\partial s}-\frac{5V}{r+s}\right)ds=\int_{s'}^{\frac{1}{2}\sigma_0}\left[\frac{(c+h)\sigma_0}{9a}\left\{\left(\frac{\sigma_0}{\frac{1}{2}\sigma_0+s}\right)^9-1\right\}-\frac{h}{a}(\tfrac{1}{2}\sigma_0-s)\right]\frac{(\frac{1}{2}\sigma_0+s')(\frac{1}{2}\sigma_0-r')}{(r'+s')^6}$$

$$\times\left[20(\tfrac{1}{2}\sigma_0+s)^3+180\frac{(\frac{1}{2}\sigma_0-r')(s-s')(\frac{1}{2}\sigma_0+s)^2}{r'+s'}\right.$$

$$\left.+420\frac{(\frac{1}{2}\sigma_0-r')^2(s-s')^2(\frac{1}{2}\sigma_0+s)}{(r'+s')^2}+280\frac{(\frac{1}{2}\sigma_0-r')^3(s-s')^3}{(r'+s')^3}\right]ds.$$

Along BC we have

$$Z=-\tfrac{1}{9}H\frac{\sigma_0}{a}\left\{\left(\frac{\sigma_0}{\frac{1}{2}\sigma_0+r}\right)^9-1\right\}+\frac{H}{a}(\tfrac{1}{2}\sigma_0-r),$$

$$\frac{\partial V}{\partial r}-\frac{5V}{r+s}=\frac{(\frac{1}{2}\sigma_0+r')(\frac{1}{2}\sigma_0-s')(\frac{1}{2}\sigma_0+r)^3}{(r'+s')^6}(20-180\zeta+420\zeta^2-280\zeta^3),$$

where

$$\zeta=-\frac{(\frac{1}{2}\sigma_0-s')(r-r')}{(r'+s')(\frac{1}{2}\sigma_0+r)},$$

so that

$$\int_{BC}Z\left(\frac{\partial V}{\partial r}-\frac{5V}{r+s}\right)dr=\int_{\frac{1}{2}\sigma_0}^{r'}\left[-\frac{H\sigma_0}{9a}\left\{\left(\frac{\sigma_0}{\frac{1}{2}\sigma_0+r}\right)^9-1\right\}+\frac{H}{a}(\tfrac{1}{2}\sigma_0-r)\right]\frac{(\frac{1}{2}\sigma_0+r')(\frac{1}{2}\sigma_0-s')}{(r'+s')^6}$$

$$\times\left[20(\tfrac{1}{2}\sigma_0+r)^3+180\frac{(\frac{1}{2}\sigma_0-s')(r-r')(\frac{1}{2}\sigma_0+r)^2}{r'+s'}\right.$$

$$\left.+420\frac{(\frac{1}{2}\sigma_0-s')^2(r-r')^2(\frac{1}{2}\sigma_0+r)}{(r'+s')^2}+280\frac{(\frac{1}{2}\sigma_0-s')^3(r-r')^3}{(r'+s')^3}\right]dr.$$

There is no difficulty in the integrations. After they have been performed, we may suppress the accents on r' and s', and so obtain the formula for Z in the first middle wave expressed as a function of r and s. We find, for example, as the coefficient of $(c+h)\sigma_0/9a$ the expression

$$\left(\frac{\sigma_0}{r+s}\right)^9-\left(\frac{\frac{1}{2}\sigma_0+s}{r+s}\right)^5-\frac{(\frac{1}{2}\sigma_0+s)(\frac{1}{2}\sigma_0-r)}{(r+s)^6}\{9\sigma_0^4-5(\tfrac{1}{2}\sigma_0+s)^4\}$$

$$-\frac{(\frac{1}{2}\sigma_0+s)(\frac{1}{2}\sigma_0-r)^2}{(r+s)^7}\{81\sigma_0^4-90\sigma_0^3(\tfrac{1}{2}\sigma_0+s)+15(\tfrac{1}{2}\sigma_0+s)^4\}$$

$$-\frac{(\frac{1}{2}\sigma_0+s)(\frac{1}{2}\sigma_0-r)^3}{(r+s)^8}\{189\sigma_0^4-420\sigma_0^3(\tfrac{1}{2}\sigma_0+s)+270\sigma_0^2(\tfrac{1}{2}\sigma_0+s)^2-35(\tfrac{1}{2}\sigma_0+s)^4\}$$

$$-\frac{(\frac{1}{2}\sigma_0+s)(\frac{1}{2}\sigma_0-r)^4}{(r+s)^9}\{126\sigma_0^4-420\sigma_0^3(\tfrac{1}{2}\sigma_0+s)+540\sigma_0^2(\tfrac{1}{2}\sigma_0+s)^2-315\sigma_0(\tfrac{1}{2}\sigma_0+s)^3$$

$$+70(\tfrac{1}{2}\sigma_0+s)^4\}.$$

In this expression we put

$$\tfrac{1}{2}\sigma_0 - r = (\tfrac{1}{2}\sigma_0 + s) - (r+s) = \tfrac{1}{2}(\sigma_0 + \sigma - u) - \sigma,$$

and find that the expression is the same as

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4 \left\{\frac{\phi_1(\sigma-u)}{\sigma}\right\},$$

where

$$\phi_1(\sigma-u) = \tfrac{1}{105}\left[\sigma_0^9 - \frac{1}{2^5}(\sigma_0+\sigma-u)^5\{126\sigma_0^4 - 210\sigma_0^3(\sigma_0+\sigma-u) + 135\sigma_0^2(\sigma_0+\sigma-u)^2 \right.$$
$$\left. - \tfrac{315}{8}\sigma_0(\sigma_0+\sigma-u)^3 + \tfrac{35}{8}(\sigma_0+\sigma-u)^4\}\right].$$

The remaining terms in the expression for Z may be treated in the same way, and we obtain finally, as the expression for Z in the first middle wave,

$$Z = \frac{(c+h)\sigma_0}{9a}\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{\phi_1(\sigma-u)}{\sigma}\right\} + \frac{h}{a}\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{\psi_1(\sigma-u)}{\sigma}\right\}$$
$$-\frac{H\sigma_0}{9a}\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{\phi_1(\sigma+u)}{\sigma}\right\} - \frac{H}{a}\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{\psi_1(\sigma+u)}{\sigma}\right\},$$

in which the expression denoted by ϕ_1 has been written down, and ψ_1 is given by the formula

$$\psi_1(\sigma-u) = -\frac{1}{15\times 2^9}(\sigma_0+\sigma-u)^5(\sigma_0-\sigma+u)^5.$$

It may be observed that the differential coefficient of the function ϕ_1 is given by the equation

$$\phi_1^{(1)}(\sigma-u) = -\frac{3}{2^8}(\sigma_0+\sigma-u)^4(\sigma_0-\sigma+u)^4.$$

Although the actual calculation of Z is rather long, it is comparatively easy to verify that the form obtained satisfies the conditions by which Z was determined.

16. *Transformation of the Formula.*—The form taken by Z in the first middle wave is

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{\Phi_1(\sigma+u)+\Psi_1(\sigma-u)}{\sigma}\right\},$$

where

$$\Phi_1(\sigma+u) = -\frac{H\sigma_0}{9a}\phi_1(\sigma+u) - \frac{H}{a}\psi_1(\sigma+u),$$

$$\Psi_1(\sigma-u) = \frac{(c+h)\sigma_0}{9a}\phi_1(\sigma-u) + \frac{h}{a}\psi_1(\sigma-u),$$

so that Φ_1 and Ψ_1 are rational integral functions of the tenth degree. Now it is important to observe that when $1 \leqslant \nu \leqslant 8$,

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{(\sigma+u)^\nu}{\sigma}\right\} = (-1)^\nu\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{(\sigma-u)^\nu}{\sigma}\right\},$$

while for $\nu = 9$ we have

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4 \left\{\frac{(\sigma+u)^9}{\sigma}\right\} = 2\,(2\,.\,4\,.\,6\,.\,8) - \left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4 \left\{\frac{(\sigma-u)^9}{\sigma}\right\},$$

and for $\nu = 10$ we have

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4 \left\{\frac{(\sigma+u)^{10}}{\sigma}\right\} = 2u\,(2\,.\,4\,.\,6\,.\,8\,.\,10) + \left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4 \left\{\frac{(\sigma-u)^{10}}{\sigma}\right\}.$$

It follows that the expression for Z can be written either in the form

$$Z = \left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4 \left\{\frac{Q_1(\sigma+u)}{\sigma}\right\} + K_1 + L_1 u,$$

or in the form

$$Z = \left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4 \left\{\frac{q_1(\sigma-u)}{\sigma}\right\} + k_1 + l_1 u,$$

where

$$K_1 = 2\,.(2\,.\,4\,.\,6\,.\,8) \times \text{ the coefficient of } (\sigma-u)^9 \text{ in } \Psi_1(\sigma-u),$$
$$L_1 = -2\,.(2\,.\,4\,.\,6\,.\,8\,.\,10) \times \text{ the coefficient of } (\sigma-u)^{10} \text{ in } \Psi_1(\sigma-u),$$
$$k_1 = 2\,.(2\,.\,4\,.\,6\,.\,8) \times \text{ the coefficient of } (\sigma+u)^9 \text{ in } \Phi_1(\sigma+u),$$
$$l_1 = 2\,.(2\,.\,4\,.\,6\,.\,8\,.\,10) \times \text{ the coefficient of } (\sigma+u)^{10} \text{ in } \Phi_1(\sigma+u),$$

and Q_1 and q_1 are certain rational integral functions of the 9th degree. The explicit expressions are

$$K_1 = -\tfrac{1}{9}(c+h)(\sigma_0/a), \quad L_1 = -h/a, \quad k_1 = \tfrac{1}{9}H\sigma_0/a, \quad l_1 = -H/a,$$

$$Q_1(\sigma+u) = \frac{\sigma_0^{10}}{1890a}(c+h-H) + \frac{h-H}{15\times 2^9 a}\{\sigma_0^2-(\sigma+u)^2\}$$
$$+\frac{\sigma_0(c+h+H)}{945\times 2^8 a}\{315\sigma_0^8(\sigma+u)-420\sigma_0^6(\sigma+u)^3+378\sigma_0^4(\sigma+u)^5$$
$$-180\sigma_0^2(\sigma+u)^7+35(\sigma+u)^9\},$$

and $q_1(\sigma-u)$ is obtained from $Q_1(\sigma+u)$ by writing $-(\sigma-u)$ for $(\sigma+u)$.

17. *Incidence of the First Middle Wave on the Pistons.*—The values of all the quantities at the piston M, at the instant when the first middle wave reaches it, are to be found from the formulæ, connected with the receding front of the wave, by putting $x_0 = 0$. We see that the receding front reaches the piston M at the time T_1, where

$$T_1 = \frac{(\frac{1}{2}c+H)^2}{aH} - \frac{H}{a},$$

that the corresponding value of σ is Σ_1, where

$$\Sigma_1 = \sigma_0\left(\frac{H}{\frac{1}{2}c+H}\right)^{1/5},$$

that the corresponding values of r, s, u are R_1, S_1, U_1, where

$$R_1 = \Sigma_1 - \tfrac{1}{2}\sigma_0, \quad S_1 = \tfrac{1}{2}\sigma_0, \quad U_1 = -(\sigma_0 - \Sigma_1),$$

and that the corresponding value of Z is Z_1, where

$$Z_1 = \frac{H}{a}\left\{\tfrac{10}{9}\sigma_0 - \Sigma_1 - \frac{\sigma_0}{9}\left(\frac{\sigma_0}{\Sigma_1}\right)^9\right\}.$$

In like manner we see that the advancing front reaches the piston m at the time t_1, where

$$t_1 = \frac{(\frac{1}{2}c+h)^2}{ah} - \frac{h}{a},$$

and that the corresponding values of σ, r, s, u, Z, are σ_1, r_1, s_1, u_1, z_1, where

$$\sigma_1 = \sigma_0\left(\frac{h}{\frac{1}{2}c+h}\right)^{1/5}, \qquad r_1 = \tfrac{1}{2}\sigma_0, \qquad s_1 = (\sigma_1 - \tfrac{1}{2}\sigma_0), \qquad u_1 = \sigma_0 - \sigma_1,$$

$$z_1 = \frac{(c+h)\,\sigma_0}{9a}\left\{\left(\frac{\sigma_0}{\sigma_1}\right)^9 - 1\right\} - \frac{h}{a}(\sigma_0 - \sigma_1).$$

The First Reflected Waves.

18. *Conditions determining the First Reflected Wave from the Left.*—After the instant $t = T_1$ the formulæ belonging to the progressive wave from the left cease to hold in the neighbourhood of $x_0 = 0$, and a new compound wave, the first reflected wave from the left, is generated there and encroaches upon the first middle wave. The junction is characterized by the value R_1 of r. The conditions determining the reflected wave are the condition which holds at the junction, where $r = R_1$, and the condition which holds at the piston, where $x_0 = 0$. It is further necessary that x_0 should vanish when $r = R_1$ and $s = S_1$.

The condition which holds at the junction is that the value of Z, calculated from the formulæ belonging to the reflected wave, should be equal to that calculated from the formulæ belonging to the first middle wave when $r = R_1$. The condition which holds at the piston is the equation of motion of the piston, viz., that at $x_0 = 0$

$$M\frac{\partial u}{\partial t} = -\omega p.$$

The condition that x_0 should vanish when $r = R_1$ and $s = S_1$ is the condition that $\partial Z/\partial\sigma$ should vanish for these values of r and s.

To express the condition which holds at the piston in terms of Z we substitute $p_0(\sigma/\sigma_0)^{11}$ for p, and

$$\frac{\partial x_0}{\partial \sigma}\Big/\left(\frac{\partial x_0}{\partial \sigma}\frac{\partial t}{\partial u} - \frac{\partial x_0}{\partial u}\frac{\partial t}{\partial \sigma}\right)$$

for $\partial u/\partial t$. Then we have

$$\frac{M}{\omega p_0}\frac{\partial x_0}{\partial \sigma} = -\left(\frac{\sigma}{\sigma_0}\right)^{11}\left(\frac{\partial x_0}{\partial \sigma}\frac{\partial t}{\partial u} - \frac{\partial x_0}{\partial u}\frac{\partial t}{\partial \sigma}\right).$$

Again we substitute $-\Pi\,\partial Z/\partial\sigma$ for x_0 and $\partial Z/\partial u$ for t, and put $\partial Z/\partial\sigma = 0$, obtaining the equation

$$\frac{\partial^2 Z}{\partial \sigma^2}\frac{\partial^2 Z}{\partial u^2} - \left(\frac{\partial^2 Z}{\partial\sigma\,\partial u}\right)^2 = -\frac{10\Pi\sigma_0^{10}}{a\sigma^{11}}\frac{\partial^2 Z}{\partial \sigma^2}.$$

The condition which holds at the junction is that

$$Z = \left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{Q_1(\sigma+u)}{\sigma}\right\} + K_1 + L_1 u$$

for all values of σ and u for which $\sigma + u = 2R_1$.

19. *Determination of the First Reflected Wave from the Left.*—These conditions can be satisfied by assuming for Z the form

$$Z = \left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{F_1(\sigma+u)}{\sigma}\right\} + K_1 + L_1 u,$$

expanding the unknown function F_1 in the series

$$F_1(\sigma+u) = A_0 + A_1(\sigma+u-2R_1) + A_2(\sigma+u-2R_1)^2 + \ldots,$$

and finding the coefficients of this series.

The condition which holds at the junction determines the coefficients $A_0, A_1, \ldots, A_4$. The condition that $\partial Z/\partial\sigma$ vanishes when $r = R_1$ and $s = S_1$ determines the coefficient A_5. The remaining coefficients are to be determined by the condition which holds at the piston.

We have

$$Z = 105\frac{F_1(\sigma+u)}{\sigma^9} - 105\frac{F_1^{(1)}(\sigma+u)}{\sigma^8} + 45\frac{F_1^{(2)}(\sigma+u)}{\sigma^7} - 10\frac{F_1^{(3)}(\sigma+u)}{\sigma^6} + \frac{F_1^{(4)}(\sigma+u)}{\sigma^5} + K_1 + L_1 u,$$

from which we find

$$A_0 = Q_1(2R_1),\ A_1 = Q_1^{(1)}(2R_1),\ 2!\,A_2 = Q_1^{(2)}(2R_1),\ 3!\,A_3 = Q_1^{(3)}(2R_1),\ 4!\,A_4 = Q_1^{(4)}(2R_1).$$

We have also

$$\frac{\partial Z}{\partial\sigma} = -945\frac{F_1(\sigma+u)}{\sigma^{10}} + 945\frac{F_1^{(1)}(\sigma+u)}{\sigma^9} - 420\frac{F_1^{(2)}(\sigma+u)}{\sigma^8} + 105\frac{F_1^{(3)}(\sigma+u)}{\sigma^7} - 15\frac{F_1^{(4)}(\sigma+u)}{\sigma^6} + \frac{F_1^{(5)}(\sigma+u)}{\sigma^5},$$

and the condition that this vanishes when $r = R_1$ and $s = S_1$ gives

$$-945\frac{A_0}{\Sigma_1^{10}} + 945\frac{A_1}{\Sigma_1^{9}} - 420\frac{2\,!\,A_2}{\Sigma_1^{8}} + 105\frac{3\,!\,A_3}{\Sigma_1^{7}} - 15\frac{4\,!\,A_4}{\Sigma_1^{6}} + \frac{5\,!\,A_5}{\Sigma_1^{5}} = 0,$$

thus determining the coefficient A_5. It is seen easily that $5\,!\,A_5 = Q_1^{(5)}(2R_1)$.

Now when $\partial Z/\partial\sigma = 0$ the differential equation for Z shows that

$$\frac{\partial^2 Z}{\partial\sigma^2} = \frac{\partial^2 Z}{\partial u^2}.$$

Also we have in general

$$\frac{\partial^2 Z}{\partial u^2} = 105\frac{F_1^{(2)}(\sigma+u)}{\sigma^9} - 105\frac{F_1^{(3)}(\sigma+u)}{\sigma^8} + 45\frac{F_1^{(4)}(\sigma+u)}{\sigma^7} - 10\frac{F_1^{(5)}(\sigma+u)}{\sigma^6} + \frac{F_1^{(6)}(\sigma+u)}{\sigma^5},$$

$$\frac{\partial^2 Z}{\partial\sigma\partial u} = -945\frac{F_1^{(1)}(\sigma+u)}{\sigma^{10}} + 945\frac{F_1^{(2)}(\sigma+u)}{\sigma^9} - 420\frac{F_1^{(3)}(\sigma+u)}{\sigma^8} + 105\frac{F_1^{(4)}(\sigma+u)}{\sigma^7} - 15\frac{F_1^{(5)}(\sigma+u)}{\sigma^6} + \frac{F_1^{(6)}(\sigma+u)}{\sigma^5},$$

and therefore when $\partial Z/\partial\sigma = 0$ we have the equation

$$\begin{aligned}&\sigma^2\{105F_1^{(2)} - 105\sigma F_1^{(3)} + 45\sigma^2 F_1^{(4)} - 10\sigma^3 F_1^{(5)} + \sigma^4 F_1^{(6)}\}^2\\ &- \{945F_1^{(1)} - 945\sigma F_1^{(2)} + 420\sigma^2 F_1^{(3)} - 105\sigma^3 F_1^{(4)} + 15\sigma^4 F_1^{(5)} - \sigma^5 F_1^{(6)}\}^2\\ &+ 10H(\sigma_0^{10}/a)\{105F_1^{(2)} - 105\sigma F_1^{(3)} + 45\sigma^2 F_1^{(4)} - 10\sigma^3 F_1^{(5)} + \sigma^4 F_1^{(6)}\} = 0\end{aligned}$$

as well as the equation

$$945F_1 - 945\sigma F_1^{(1)} + 420\sigma^2 F_1^{(2)} - 105\sigma^3 F_1^{(3)} + 15\sigma^4 F_1^{(4)} - \sigma^5 F_1^{(5)} = 0.$$

The equation expressing the condition which holds at the piston is linear in $F_1^{(6)}$, and therefore can be solved for $F_1^{(6)}$ without ambiguity. As it holds for $r = R_1$ and $s = S_1$, it determines the coefficient A_6. The equation in question holds for all values of σ and u for which the equation expressing the vanishing of $\partial Z/\partial\sigma$ holds, and it can therefore be differentiated totally with respect to σ, u being treated as a function of σ in accordance with the equation $\partial Z/\partial\sigma = 0$. This process yields an equation which determines the coefficient A_7 without ambiguity. A second differentiation yields an equation from which the value of the coefficient A_8 may be found. By proceeding in this way we may obtain as many of the coefficients A as we wish.

This method of determining the coefficients A_6, A_7, ... is not very well adapted to numerical computation, and other methods will be explained presently.

20. *Determination of the First Reflected Wave from the Right.*—The junction of the reflected wave and the first middle wave is characterised by the value s_1 of s. The conditions determining the reflected wave are the condition which holds at the junction, where $s = s_1$, and the condition which holds at the piston, where $x_0 = c$. Further, x_0 must be equal to c when $r = r_1$ and $s = s_1$.

At the junction, where $s = s_1$, the value of Z must be the same whether it is found from the formulæ belonging to the reflected wave or from those belonging to the first middle wave. At $x_0 = c$ the equation of motion of the piston, viz., the equation

$$m \frac{\partial u}{\partial t} = \omega p$$

must hold. The equation $-\Pi\, \partial Z/\partial \sigma = c$ must hold at $r = r_1$ and $s = s_1$.

To express these conditions it is convenient to write

$$Z = \left(\frac{1}{\sigma}\frac{\partial}{\partial \sigma}\right)^4 \left(\frac{c\sigma_0^{10}}{945a\sigma}\right) + Z' ;$$

then Z' satisfies the same differential equation as Z, and $\partial Z'/\partial \sigma$ vanishes when $x_0 = c$.

The condition which holds at the junction is that

$$Z = \left(\frac{1}{\sigma}\frac{\partial}{\partial \sigma}\right)^4 \left\{\frac{q_1(\sigma - u)}{\sigma}\right\} + k_1 + l_1 u$$

for all values of σ and u for which $\sigma - u = 2s_1$.

The condition which holds at the piston is that

$$\frac{\partial^2 Z'}{\partial \sigma^2}\frac{\partial^2 Z'}{\partial u^2} - \left(\frac{\partial^2 Z'}{\partial \sigma\, \partial u}\right)^2 = \frac{10h\sigma_0^{10}}{a\sigma^{11}}\frac{\partial^2 Z'}{\partial \sigma^2}$$

when $\partial Z'/\partial \sigma = 0$.

These conditions can be satisfied by assuming for Z the form

$$Z = \left(\frac{1}{\sigma}\frac{\partial}{\partial \sigma}\right)^4 \left[\frac{1}{\sigma}\left\{\frac{c\sigma_0^{10}}{945a} + f_1(\sigma - u)\right\}\right] + k_1 + l_1 u,$$

expanding the unknown function f_1 in the series

$$f_1 = a_0 + a_1(\sigma - u - 2s_1) + a_2(\sigma - u - 2s_1)^2 + \dots,$$

and finding the coefficients of this series.

For the coefficients $a_0, a_1, \dots, a_4$ we find

$$a_0 + \frac{c\sigma_0^{10}}{945a} = q_1(2s_1),\ a_1 = q_1^{(1)}(2s_1),\ 2!\,a_2 = q_1^{(2)}(2s_1),\ 3!\,a_3 = q_1^{(3)}(2s_1),\ 4!\,a_4 = q_1^{(4)}(2s_1).$$

The coefficient a_5 is given by the equation

$$-945\frac{a_0}{\sigma_1^{10}} + 945\frac{a_1}{\sigma_1^9} - 420\frac{2!\,a_2}{\sigma_1^8} + 105\frac{3!\,a_3}{\sigma_1^7} - 15\frac{4!\,a_4}{\sigma_1^6} + \frac{5!\,a_5}{\sigma_1^5} = 0.$$

The remaining coefficients can be determined from the condition which holds at the piston in the same way as the corresponding coefficients in the formula belonging to the first reflected wave from the left could be determined.

21. *Relation between Pressure and Velocity at a Piston.*—The equation in terms of σ and u, which holds for the first reflected wave from the left at $x_0 = 0$, is the relation between the pressure on the piston M and its velocity during the time that the wave is being generated. It may also be interpreted as the equation of a certain locus in the plane of r and s. This locus passes through the point (R_1, S_1), and we may take its equation to be of the form

$$r - R_1 = B_1(s - S_1) + (B_2/\Sigma_1)(s - S_1)^2 + (B_3/\Sigma_1^2)(s - S_1)^3 + \dots .$$

Now if the coefficients B were known, we could determine x_0, as a function of r and s, from the known value, zero, of the function along the locus and the values of its differential coefficients along the same curve. These differential coefficients also are known along the locus. To prove this and obtain formulæ for these differential coefficients, we write X for x_0 and observe that the equations of Article 6 show that X satisfies the differential equation

$$\frac{\partial}{\partial u}\left(\frac{1}{\Pi}\frac{\partial X}{\partial u}\right) = \frac{\partial}{\partial \sigma}\left(\frac{1}{\Pi}\frac{\partial X}{\partial \sigma}\right),$$

which can be written either in the form

$$\frac{\partial^2 X}{\partial \sigma^2} - \frac{10}{\sigma}\frac{\partial X}{\partial \sigma} - \frac{\partial^2 X}{\partial u^2} = 0,$$

or in the form

$$\frac{\partial^2 X}{\partial r\,\partial s} - \frac{5}{r+s}\left(\frac{\partial X}{\partial r} + \frac{\partial X}{\partial s}\right) = 0.$$

Further at $x_0 = 0$ we have

$$\frac{\partial u}{\partial t} = -\frac{\omega p}{M} = -\frac{\omega p_0}{M}\left(\frac{\sigma}{\sigma_0}\right)^{11} = -\frac{a\sigma_0}{10H}\left(\frac{\sigma}{\sigma_0}\right)^{11},$$

and

$$\frac{\partial u}{\partial t} = \frac{\dfrac{\partial x_0}{\partial \sigma}}{\dfrac{\partial x_0}{\partial \sigma}\dfrac{\partial t}{\partial u} - \dfrac{\partial x_0}{\partial u}\dfrac{\partial t}{\partial \sigma}} = \frac{\Pi\dfrac{\partial x_0}{\partial \sigma}}{-\left(\dfrac{\partial x_0}{\partial \sigma}\right)^2 + \left(\dfrac{\partial x_0}{\partial u}\right)^2} = \frac{a\left(\dfrac{\sigma}{\sigma_0}\right)^{10}\dfrac{\partial x_0}{\partial \sigma}}{-\left(\dfrac{\partial x_0}{\partial \sigma}\right)^2\left\{1 - \left(\dfrac{d\sigma}{du}\right)^2\right\}},$$

where $d\sigma/du$ is to be found from the equation connecting r and s. Thus we have along this locus

$$\frac{\partial x_0}{\partial \sigma} = \frac{10H}{\sigma\left\{1 - \left(\dfrac{d\sigma}{du}\right)^2\right\}}, \qquad \frac{\partial x_0}{\partial u} = -\frac{10H\dfrac{d\sigma}{du}}{\sigma\left\{1 - \left(\dfrac{d\sigma}{du}\right)^2\right\}}.$$

The equation for X is similar in form to that for Z, and may be solved by RIEMANN's method. When this is done the coefficients B in the equation of the locus may be determined by identifying the values of X at $r = R_1$ with those given for x_0 by the formulæ for the first middle wave.

22. *Integration of the Equation for X.*—We write the equation

$$\frac{\partial^2 X}{\partial r\,\partial s} - \frac{5}{r+s}\left(\frac{\partial X}{\partial r} + \frac{\partial X}{\partial s}\right) = 0,$$

and consider also a function Y which satisfies the adjoint equation

$$\frac{\partial^2 Y}{\partial r\,\partial s} + 5\left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s}\right)\left(\frac{Y}{r+s}\right) = 0.$$

Then the integral

$$\iint\left[\frac{\partial}{\partial r}\left\{Y\left(\frac{\partial X}{\partial s} - \frac{5X}{r+s}\right)\right\} - \frac{\partial}{\partial s}\left\{X\left(\frac{\partial Y}{\partial r} + \frac{5Y}{r+s}\right)\right\}\right] dr\,ds$$

taken over any area in the plane of (r, s) vanishes, and therefore the integral

$$\int Y\left(\frac{\partial X}{\partial s} - \frac{5X}{r+s}\right) ds + X\left(\frac{\partial Y}{\partial r} + \frac{5Y}{r+s}\right) dr$$

taken round the boundary of the area also vanishes.

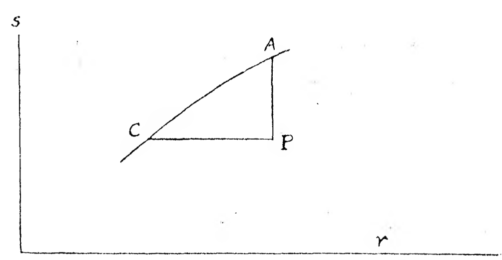


Fig. 3.

We take the area of integration to be bounded by an arc of the locus along which $X = 0$, and two lines parallel to the axes of s and r and meeting at the point P, where $r = r'$ and $s = s'$. Let these be the lines PA and PC. Then we have

$$[YX]_A - [YX]_P - \int_{PA} X\left(\frac{\partial Y}{\partial s} + \frac{5Y}{r+s}\right) ds + \int_{AC} Y\left(\frac{\partial X}{\partial s} - \frac{5X}{r+s}\right) ds + X\left(\frac{\partial Y}{\partial r} + \frac{5Y}{r+s}\right) dr$$
$$+ \int_{CP} X\left(\frac{\partial Y}{\partial r} + \frac{5Y}{r+s}\right) dr = 0,$$

or, since $X = 0$ on the arc AC,

$$[YX]_P = \int_{AC} Y\frac{\partial X}{\partial s}\,ds - \int_{PA} X\left(\frac{\partial Y}{\partial s} + \frac{5Y}{r+s}\right) ds + \int_{CP} X\left(\frac{\partial Y}{\partial r} + \frac{5Y}{r+s}\right) dr.$$

We choose Y so that, at P, $Y = 1$, along PA, where $r = r'$, $\partial Y/\partial s = -5Y/(r+s)$, and along CP, where $s = s'$, $\partial Y/\partial r = -5Y/(r+s)$. Then the value of X at P is

$$\int_{AC} Y\frac{\partial X}{\partial s}\,ds.$$

The requisite form of Y is

$$Y = \left(\frac{r'+s'}{r+s}\right)^5 F(6, -5, 1, \zeta),$$

where

$$\zeta = -\frac{(r-r')(s-s')}{(r'+s')(r+s)},$$

and we have

$$X(r', s') = \int_{AC} Y\left(\frac{\partial X}{\partial \sigma} - \frac{\partial X}{\partial u}\right)\tfrac{1}{2}(d\sigma - du)$$

$$= \int_{AC} Y\frac{10H}{\sigma}\frac{1+\dfrac{d\sigma}{du}}{1-\left(\dfrac{d\sigma}{du}\right)^2}\tfrac{1}{2}\left(\frac{d\sigma}{du}-1\right)du$$

$$= -5H\int_{AC}\frac{Y}{\sigma}\,du.$$

23. *Determination of the Coefficients B.*—The integral $\int_{AC}(Y/\sigma)\,du$ may be evaluated approximately by assuming, as in Article 21, that the equation of the locus, of which AC is an arc, is of the form

$$(r-R_1)/\Sigma_1 = B_1\delta + B_2\delta^2 + B_3\delta^3 + \ldots,$$

where δ stands for $(s - S_1)/\Sigma_1$. Then along AC we have

$$u - U_1 = \Sigma_1\{(B_1-1)\delta + B_2\delta^2 + B_3\delta^3 + \ldots\},$$

$$du = \Sigma_1\{(B_1-1) + 2B_2\delta + 3B_3\delta^2 + \ldots\}\,d\delta,$$

$$\frac{Y}{\sigma} = \frac{(r'+s')^5}{\sigma^6}\left\{1 + 30\,\frac{(r-r')(s-s')}{(r'+s')\,\sigma} + 210\,\frac{(r-r')^2(s-s')^2}{(r'+s')^2\,\sigma^2} + \ldots\right\}.$$

Also any inverse power of σ can be expanded in powers of δ by means of the equations

$$\sigma = \Sigma_1\{1 + (B_1+1)\delta + B_2\delta^2 + B_3\delta^3 + \ldots\},$$

$$\sigma^{-\kappa} = \Sigma_1^{-\kappa}\left\{1 - \kappa\frac{\sigma-\Sigma_1}{\Sigma_1} + \frac{\kappa(\kappa+1)}{2!}\left(\frac{\sigma-\Sigma_1}{\Sigma_1}\right)^2 - \ldots\right\},$$

which give

$$\begin{aligned}\sigma^{-\kappa} = \Sigma_1^{-\kappa}\Big[&1 - \kappa(B_1+1)\delta - \left\{\kappa B_2 - \frac{\kappa(\kappa+1)}{2!}(B_1+1)^2\right\}\delta^2 \\ &- \left\{\kappa B_3 - \kappa(\kappa+1)(B_1+1)B_2 + \frac{\kappa(\kappa+1)(\kappa+2)}{3!}(B_1+1)^3\right\}\delta^3 \\ &- \Big\{\kappa B_4 - \frac{\kappa(\kappa+1)}{2!}B_2^2 - \kappa(\kappa+1)(B_1+1)B_3 + \frac{\kappa(\kappa+1)(\kappa+2)}{2!}(B_1+1)^2 B_2 \\ &- \frac{\kappa(\kappa+1)(\kappa+2)(\kappa+3)}{4!}(B_1+1)^4\Big\}\delta^4 - \ldots\Big].\end{aligned}$$

To obtain the expression for X on $r = R_1$ we have to put R_1 for r', so that

$$r - r' = \Sigma_1(B_1\delta + B_2\delta^2 + B_3\delta^3 + \ldots),$$

and for $s-s'$ we have to put $\Sigma_1(\delta-\delta')$, where δ' stands for $(s'-S_1)/\Sigma_1$. If the expansions are carried as far as the fourth order, the result is that, to the fifth order in δ',

$$\begin{aligned}\int_{AC}(Y/\sigma)\,du = (B_1-1)\,\delta' + \{B_2-(3B_1-2)(B_1-1)\}\,\delta'^2 \\ + \{B_3-3B_2(2B_1-1)+(7B_1^2-6B_1+2)(B_1-1)\}\,\delta'^3 \\ + \{B_4-2B_3(3B_1-1)-3B_2^2+\tfrac{1}{2}B_2(42B_1^2-33B_1+5) \\ -\tfrac{1}{2}(28B_1^3-21B_1^2+9B_1-2)(B_1-1)\}\,\delta'^4 \\ + \{B_5-\tfrac{1}{5}B_4(30B_1-7)-6B_2B_3+\tfrac{1}{10}B_3(210B_1^2-108B_1+7) \\ +\tfrac{1}{5}B_2^2(105B_1-27)-\tfrac{1}{10}B_2(560B_1^3-462B_1^2+81B_1-7) \\ +\tfrac{1}{5}(126B_1^4-56B_1^3+21B_1^2-6B_1+1)(B_1-1)\}\,\delta'^5.\end{aligned}$$

Now at any point (R_1, s') on $r = R_1$ the formula for the first middle wave gives

$$\begin{aligned}x_0 &= -a\left(\frac{\sigma}{\sigma_0}\right)^{10}\frac{\partial}{\partial\sigma}\left[\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{Q_1(\sigma+u)}{\sigma}\right\}+K_1+L_1u\right] \\ &= -\frac{a}{\sigma_0^{10}}\{-945Q_1(2R_1)+945\sigma Q_1^{(1)}(2R_1)-420\sigma^2Q_1^{(2)}(2R_1)+105\sigma^3Q_1^{(3)}(2R_1) \\ &\qquad -15\sigma^4Q_1^{(4)}(2R_1)+\sigma^5Q_1^{(5)}(2R_1)\},\end{aligned}$$

in which we have to put $\sigma = \Sigma_1(1+\delta')$. Then, since x_0 vanishes with δ', we have without any approximation

$$\begin{aligned}x_0 = -\frac{a}{\sigma_0^{10}}[&\{945\Sigma_1Q_1^{(1)}(2R_1)-840\Sigma_1^2Q_1^{(2)}(2R_1)+315\Sigma_1^3Q_1^{(3)}(2R_1)-60\Sigma_1^4Q_1^{(4)}(2R_1) \\ &+5\Sigma_1^5Q_1^{(5)}(2R_1)\}\,\delta' \\ &+\{-420\Sigma_1^2Q_1^{(2)}(2R_1)+315\Sigma_1^3Q_1^{(3)}(2R_1)-90\Sigma_1^4Q_1^{(4)}(2R_1)+10\Sigma_1^5Q_1^{(5)}(2R_1)\}\,\delta'^2 \\ &+\{105\Sigma_1^3Q_1^{(3)}(2R_1)-60\Sigma_1^4Q_1^{(4)}(2R_1)+10\Sigma_1^5Q_1^{(5)}(2R_1)\}\,\delta'^3 \\ &+\{-15\Sigma_1^4Q_1^{(4)}(2R_1)+5\Sigma_1^5Q_1^{(5)}(2R_1)\}\,\delta'^4 \\ &+\Sigma_1^5Q_1^{(5)}(2R_1)\,\delta'^5].\end{aligned}$$

The coefficients $B_1, \ldots, B_5$ can be determined successively by equating the coefficients of powers of δ' in the expressions for $\int_{AC}(Y/\sigma)\,du$ and $-x_0/5H$. If additional coefficients $B_6, \ldots$, are desired, they may be found by equating to zero the coefficients of powers of δ' higher than the fifth in the expansion of $\int_{AC}(Y/\sigma)\,du$.

The expansion of $x_0(R_1, s')$ in powers of δ' may, of course, be found from the expression for Z in the first middle wave without transformation to the Q form. In particular it may be proved that $B_1 = 6-4\Sigma_1/\sigma_0$.

24. *Second Method of determining the Coefficients A.*—When the coefficients B are known, the coefficients $A_6, \ldots$ may be found in the following way.

Since x_0, which is $-a(\sigma/\sigma_0)^{10}\,\partial Z/\partial\sigma$, vanishes at all points of the locus

$$r-R_1 = B_1(s-S_1)+(B_2/\Sigma_1)(s-S_1)^2+(B_3/\Sigma_1^2)(s-S_1)^3+\ldots,$$

the expression

$$945F_1(2r)-945\sigma F_1^{(1)}(2r)+420\sigma^2F_1^{(2)}(2r)-105\sigma^3F_1^{(3)}(2r)+15\sigma^4F_1^{(4)}(2r)-\sigma^5F_1^{(5)}(2r),$$

in which

$$F_1(2r) = A_0+A_1(2r-2R_1)+A_2(2r-2R_1)^2+\ldots,$$

must become identically zero on substitution of

$$\Sigma_1(B_1\delta+B_2\delta^2+B_3\delta^3+\ldots)$$

for $r-R_1$ and of

$$\Sigma_1\{1+(B_1+1)\delta+B_2\delta^2+B_3\delta^3+\ldots\}$$

for σ. Now the powers of σ/Σ_1 and $(r-R_1)/\Sigma_1$ can all be expanded in powers of δ, and then the coefficients of the powers of δ in the expansion of

$$945\frac{F_1(2r)}{\Sigma_1^{10}}-945\frac{\sigma}{\Sigma_1}\frac{F_1^{(1)}(2r)}{\Sigma_1^{9}}+420\left(\frac{\sigma}{\Sigma_1}\right)^2\frac{F_1^{(2)}(2r)}{\Sigma_1^{8}}-105\left(\frac{\sigma}{\Sigma_1}\right)^3\frac{F_1^{(3)}(2r)}{\Sigma_1^{7}}$$
$$+15\left(\frac{\sigma}{\Sigma_1}\right)^4\frac{F_1^{(4)}(2r)}{\Sigma_1^{6}}-\left(\frac{\sigma}{\Sigma_1}\right)^5\frac{F_1^{(5)}(2r)}{\Sigma_1^{5}}$$

can be equated severally to zero. The equations thus arising give the values of A_6, A_7, ..., successively. Suppressing the algebra, which is rather long, we may write down the results in the following form:—

The equation for A_6 is

$$2B_1\frac{6!\,A_6}{\Sigma_1^4} = 945(B_1-1)\frac{A_1}{\Sigma_1^9}-210(5B_1-4)\frac{2!\,A_2}{\Sigma_1^8}+105(5B_1-3)\frac{3!\,A_3}{\Sigma_1^7}$$
$$-30(5B_1-2)\frac{4!\,A_4}{\Sigma_1^6}+5(5B_1-1)\frac{5!\,A_5}{\Sigma_1^5}.$$

The equation for A_7 is

$$2B_1^2\frac{7!\,A_7}{\Sigma_1^3} = 2B_1^2\left(945\frac{2!\,A_2}{\Sigma_1^8}-945\frac{3!\,A_3}{\Sigma_1^7}+420\frac{4!\,A_4}{\Sigma_1^6}-105\frac{5!\,A_5}{\Sigma_1^5}+15\frac{6!\,A_6}{\Sigma_1^4}\right)$$
$$+2B_1(B_1+1)\left(-945\frac{2!\,A_2}{\Sigma_1^8}+840\frac{3!\,A_3}{\Sigma_1^7}-315\frac{4!\,A_4}{\Sigma_1^6}+60\frac{5!\,A_5}{\Sigma_1^5}-5\frac{6!\,A_6}{\Sigma_1^4}\right)$$
$$+(B_1+1)^2\left(420\frac{2!\,A_2}{\Sigma_1^8}-315\frac{3!\,A_3}{\Sigma_1^7}+90\frac{4!\,A_4}{\Sigma_1^6}-10\frac{5!\,A_5}{\Sigma_1^5}\right)$$
$$+B_2\left(945\frac{A_1}{\Sigma_1^9}-1050\frac{2!\,A_2}{\Sigma_1^8}+525\frac{3!\,A_3}{\Sigma_1^7}-150\frac{4!\,A_4}{\Sigma_1^6}+25\frac{5!\,A_5}{\Sigma_1^5}-2\frac{6!\,A_6}{\Sigma_1^4}\right).$$

The equation for A_8 is

$$
\begin{aligned}
4B_1^3 \frac{8!\,A_8}{\Sigma_1^2} = {} & 4B_1^3\left(945 \frac{3!\,A_3}{\Sigma_1^7} - 945 \frac{4!\,A_4}{\Sigma_1^6} + 420 \frac{5!\,A_5}{\Sigma_1^5} - 105 \frac{6!\,A_6}{\Sigma_1^4} + 15 \frac{7!\,A_7}{\Sigma_1^3}\right) \\
& + 6B_1^2 (B_1+1) \left(-945 \frac{3!\,A_3}{\Sigma_1^7} + 840 \frac{4!\,A_4}{\Sigma_1^6} - 315 \frac{5!\,A_5}{\Sigma_1^5} + 60 \frac{6!\,A_6}{\Sigma_1^4} - 5 \frac{7!\,A_7}{\Sigma_1^3}\right) \\
& + 6B_1 (B_1+1)^2 \left(420 \frac{3!\,A_3}{\Sigma_1^7} - 315 \frac{4!\,A_4}{\Sigma_1^6} + 90 \frac{5!\,A_5}{\Sigma_1^5} - 10 \frac{6!\,A_6}{\Sigma_1^4}\right) \\
& + 3 (B_1+1)^3 \left(-105 \frac{3!\,A_3}{\Sigma_1^7} + 60 \frac{4!\,A_4}{\Sigma_1^6} - 10 \frac{5!\,A_5}{\Sigma_1^5}\right) \\
& + 6B_1B_2\left(945 \frac{2!\,A_2}{\Sigma_1^8} - 1050 \frac{3!\,A_3}{\Sigma_1^7} + 525 \frac{4!\,A_4}{\Sigma_1^6} - 150 \frac{5!\,A_5}{\Sigma_1^5} + 25 \frac{6!\,A_6}{\Sigma_1^4} - 2 \frac{7!\,A_7}{\Sigma_1^3}\right) \\
& + 3 (B_1+1) B_2\left(-1050 \frac{2!\,A_2}{\Sigma_1^8} + 1050 \frac{3!\,A_3}{\Sigma_1^7} - 450 \frac{4!\,A_4}{\Sigma_1^6} + 100 \frac{5!\,A_5}{\Sigma_1^5} - 10 \frac{6!\,A_6}{\Sigma_1^4}\right) \\
& + 3B_3\left(945 \frac{A_1}{\Sigma_1^9} - 1050 \frac{2!\,A_2}{\Sigma_1^8} + 525 \frac{3!\,A_3}{\Sigma_1^7} - 150 \frac{4!\,A_4}{\Sigma_1^6} + 25 \frac{5!\,A_5}{\Sigma_1^5} - 2 \frac{6!\,A_6}{\Sigma_1^4}\right).
\end{aligned}
$$

The equation for A_9 is

$$
\begin{aligned}
2B_1^4 \frac{9!\,A_9}{\Sigma_1} = {} & 2B_1^4 \left(945 \frac{4!\,A_4}{\Sigma_1^6} - 945 \frac{5!\,A_5}{\Sigma_1^5} + 420 \frac{6!\,A_6}{\Sigma_1^4} - 105 \frac{7!\,A_7}{\Sigma_1^3} + 15 \frac{8!\,A_8}{\Sigma_1^2}\right) \\
& + 4B_1^3 (B_1+1) \left(-945 \frac{4!\,A_4}{\Sigma_1^6} + 840 \frac{5!\,A_5}{\Sigma_1^5} - 315 \frac{6!\,A_6}{\Sigma_1^4} + 60 \frac{7!\,A_7}{\Sigma_1^3} - 5 \frac{8!\,A_8}{\Sigma_1^2}\right) \\
& + 6B_1^2 (B_1+1)^2 \left(420 \frac{4!\,A_4}{\Sigma_1^6} - 315 \frac{5!\,A_5}{\Sigma_1^5} + 90 \frac{6!\,A_6}{\Sigma_1^4} - 10 \frac{7!\,A_7}{\Sigma_1^3}\right) \\
& + 6B_1 (B_1+1)^3 \left(-105 \frac{4!\,A_4}{\Sigma_1^6} + 60 \frac{5!\,A_5}{\Sigma_1^5} - 10 \frac{6!\,A_6}{\Sigma_1^4}\right) \\
& + 6 (B_1+1)^4 \left(15 \frac{4!\,A_4}{\Sigma_1^6} - 5 \frac{5!\,A_5}{\Sigma_1^5}\right) \\
& + 12B_1^2B_2 \left(945 \frac{3!\,A_3}{\Sigma_1^7} - 945 \frac{4!\,A_4}{\Sigma_1^6} + 420 \frac{5!\,A_5}{\Sigma_1^5} - 105 \frac{6!\,A_6}{\Sigma_1^4} + 15 \frac{7!\,A_7}{\Sigma_1^3} - \frac{8!\,A_8}{\Sigma_1^2}\right) \\
& + 6B_1 (3B_1+2) B_2 \left(-945 \frac{3!\,A_3}{\Sigma_1^7} + 840 \frac{4!\,A_4}{\Sigma_1^6} - 315 \frac{5!\,A_5}{\Sigma_1^5} + 60 \frac{6!\,A_6}{\Sigma_1^4} - 5 \frac{7!\,A_7}{\Sigma_1^3}\right) \\
& + 6 (B_1+1) (3B_1+1) B_2 \left(420 \frac{3!\,A_3}{\Sigma_1^7} - 315 \frac{4!\,A_4}{\Sigma_1^6} + 90 \frac{5!\,A_5}{\Sigma_1^5} - 10 \frac{6!\,A_6}{\Sigma_1^4}\right) \\
& + 3 (B_1+1)^2 B_2 \left(-315 \frac{3!\,A_3}{\Sigma_1^7} + 180 \frac{4!\,A_4}{\Sigma_1^6} - 30 \frac{5!\,A_5}{\Sigma_1^5}\right) \\
& + 3B_2^2 \left(420 \frac{2!\,A_2}{\Sigma_1^8} - 525 \frac{3!\,A_3}{\Sigma_1^7} + 300 \frac{4!\,A_4}{\Sigma_1^6} - 100 \frac{5!\,A_5}{\Sigma_1^5} + 20 \frac{6!\,A_6}{\Sigma_1^4} - 2 \frac{7!\,A_7}{\Sigma_1^3}\right) \\
& + 6B_1B_3 \left(945 \frac{2!\,A_2}{\Sigma_1^8} - 1050 \frac{3!\,A_3}{\Sigma_1^7} + 525 \frac{4!\,A_4}{\Sigma_1^6} - 150 \frac{5!\,A_5}{\Sigma_1^5} + 25 \frac{6!\,A_6}{\Sigma_1^4} - 2 \frac{7!\,A_7}{\Sigma_1^3}\right) \\
& + 3(B_1+1)B_3\left(-1050 \frac{2!\,A_2}{\Sigma_1^8} + 1050 \frac{3!\,A_3}{\Sigma_1^7} - 450 \frac{4!\,A_4}{\Sigma_1^6} + 100 \frac{5!\,A_5}{\Sigma_1^5} - 10 \frac{6!\,A_6}{\Sigma_1^4}\right) \\
& + 3B_4\left(945 \frac{A_1}{\Sigma_1^9} - 1050 \frac{2!\,A_2}{\Sigma_1^8} + 525 \frac{3!\,A_3}{\Sigma_1^7} - 150 \frac{4!\,A_4}{\Sigma_1^6} + 25 \frac{5!\,A_5}{\Sigma_1^5} - 2 \frac{6!\,A_6}{\Sigma_1^4}\right).
\end{aligned}
$$

The method avails for the calculation of as many coefficients as may be desired.

25. *Third Method of determining the Coefficients A.*—Another nearly equally effective process for finding the coefficients A_6, A_7, ..., is founded upon an expression for t, valid at the piston M.

The equation of motion of the piston shows that at $x_0 = 0$,

$$\frac{\partial u}{\partial t} = -\frac{a\sigma^{11}}{10H\sigma_0^{10}},$$

and the differential of u is always

$$\frac{\partial u}{\partial x_0}\,dx_0 + \frac{\partial u}{\partial t}\,dt,$$

so that, at $x_0 = 0$, t can be expressed as a function of s by the equation

$$t - T_1 = -\frac{10H\sigma_0^{10}}{a\Sigma_1^{11}}\int_{S_1}^{s}\left(\frac{\sigma}{\Sigma_1}\right)^{-11}\{(B_1-1)+2B_2\delta+3B_3\delta^2+\dots\}\,ds,$$

and thus $t-T_1$ can be expanded in powers of δ or $(s-S_1)/\Sigma_1$. Also, since $t = \partial Z/\partial u$, and T_1 is the value of t given by putting $r = R_1$ and $s = S_1$ in the formula for the first reflected wave from the left, we have

$$t - T_1 = 105\left\{\frac{F_1^{(1)}(2r)}{\sigma^9} - \frac{F_1^{(1)}(2R_1)}{\Sigma_1^{9}}\right\} - 105\left\{\frac{F_1^{(2)}(2r)}{\sigma^8} - \frac{F_1^{(2)}(2R_1)}{\Sigma_1^{8}}\right\}$$
$$+45\left\{\frac{F_1^{(3)}(2r)}{\sigma^7} - \frac{F_1^{(3)}(2R_1)}{\Sigma_1^{7}}\right\} - 10\left\{\frac{F_1^{(4)}(2r)}{\sigma^6} - \frac{F_1^{(4)}(2R_1)}{\Sigma_1^{6}}\right\} + \left\{\frac{F_1^{(5)}(2r)}{\sigma^5} - \frac{F_1^{(5)}(2R_1)}{\Sigma_1^{5}}\right\},$$

so that a different form of expansion can be obtained for $t-T_1$. By equating coefficients of different powers of δ in the two forms of expansion we obtain again a series of equations giving the values of A_6, A_7, ..., successively. The results may be recorded as follows:—

The equation for A_6 is

$$105\left\{-9(B_1+1)\frac{A_1}{\Sigma_1^{9}} + 2B_1\frac{2!\,A_2}{\Sigma_1^{8}}\right\} - 105\left\{-8(B_1+1)\frac{2!\,A_2}{\Sigma_1^{8}} + 2B_1\frac{3!\,A_3}{\Sigma_1^{7}}\right\}$$
$$+45\left\{-7(B_1+1)\frac{3!\,A_3}{\Sigma_1^{7}} + 2B_1\frac{4!\,A_4}{\Sigma_1^{6}}\right\} - 10\left\{-6(B_1+1)\frac{4!\,A_4}{\Sigma_1^{6}} + 2B_1\frac{5!\,A_5}{\Sigma_1^{5}}\right\}$$
$$+\left\{-5(B_1+1)\frac{5!\,A_5}{\Sigma_1^{5}} + 2B_1\frac{6!\,A_6}{\Sigma_1^{4}}\right\} = -10\frac{H}{a}\left(\frac{\sigma_0}{\Sigma_1}\right)^{10}(B_1-1).$$

The equation for A_7 is

$$105\left[\{-9B_2+45(B_1+1)^2\}\frac{A_1}{\Sigma_1^{\,9}}+2\{B_2-9B_1(B_1+1)\}\frac{2!\,A_2}{\Sigma_1^{\,8}}+\frac{2^2}{2!}B_1^{\,2}\frac{3!\,A_3}{\Sigma_1^{\,7}}\right]$$
$$-105\left[\{-8B_2+36(B_1+1)^2\}\frac{2!\,A_2}{\Sigma_1^{\,8}}+2\{B_2-8B_1(B_1+1)\}\frac{3!\,A_3}{\Sigma_1^{\,7}}+\frac{2^2}{2!}B_1^{\,2}\frac{4!\,A_4}{\Sigma_1^{\,6}}\right]$$
$$+45\left[\{-7B_2+28(B_1+1)^2\}\frac{3!\,A_3}{\Sigma_1^{\,7}}+2\{B_2-7B_1(B_1+1)\}\frac{4!\,A_4}{\Sigma_1^{\,6}}+\frac{2^2}{2!}B_1^{\,2}\frac{5!\,A_5}{\Sigma_1^{\,5}}\right]$$
$$-10\left[\{-6B_2+21(B_1+1)^2\}\frac{4!\,A_4}{\Sigma_1^{\,6}}+2\{B_2-6B_1(B_1+1)\}\frac{5!\,A_5}{\Sigma_1^{\,5}}+\frac{2^2}{2!}B_1^{\,2}\frac{6!\,A_6}{\Sigma_1^{\,4}}\right]$$
$$+\left[\{-5B_2+15(B_1+1)^2\}\frac{5!\,A_5}{\Sigma_1^{\,5}}+2\{B_2-5B_1(B_1+1)\}\frac{6!\,A_6}{\Sigma_1^{\,4}}+\frac{2^2}{2!}B_1^{\,2}\frac{7!\,A_7}{\Sigma_1^{\,3}}\right]$$
$$=-10\frac{H}{a}\left(\frac{\sigma_0}{\Sigma_1}\right)^{10}\{B_2-\tfrac{11}{2}(B_1+1)(B_1-1)\}.$$

The equation for A_8 is

$$105\Big[\{-9B_3+90(B_1+1)B_2-165(B_1+1)^3\}\frac{A_1}{\Sigma_1^{\,9}}+2\{B_3-9(2B_1+1)B_2+45B_1(B_1+1)^2\}\frac{2!\,A_2}{\Sigma_1^{\,8}}$$
$$+\frac{2^2}{2!}\{2B_1B_2-9B_1^{\,2}(B_1+1)\}\frac{3!\,A_3}{\Sigma_1^{\,7}}+\frac{2^3}{3!}B_1^{\,3}\frac{4!\,A_4}{\Sigma_1^{\,6}}\Big]$$
$$-105\Big[\{-8B_3+72(B_1+1)B_2-120(B_1+1)^3\}\frac{2!\,A_2}{\Sigma_1^{\,8}}$$
$$+2\{B_3-8(2B_1+1)B_2+36B_1(B_1+1)^2\}\frac{3!\,A_3}{\Sigma_1^{\,7}}$$
$$+\frac{2^2}{2!}\{2B_1B_2-8B_1^{\,2}(B_1+1)\}\frac{4!\,A_4}{\Sigma_1^{\,6}}+\frac{2^3}{3!}B_1^{\,3}\frac{5!\,A_5}{\Sigma_1^{\,5}}\Big]$$
$$+45\Big[\{-7B_3+56(B_1+1)B_2-84(B_1+1)^3\}\frac{3!\,A_3}{\Sigma_1^{\,7}}$$
$$+2\{B_3-7(2B_1+1)B_2+28B_1(B_1+1)^2\}\frac{4!\,A_4}{\Sigma_1^{\,6}}$$
$$+\frac{2^2}{2!}\{2B_1B_2-7B_1^{\,2}(B_1+1)\}\frac{5!\,A_5}{\Sigma_1^{\,5}}+\frac{2^3}{3!}B_1^{\,3}\frac{6!\,A_6}{\Sigma_1^{\,4}}\Big]$$
$$-10\Big[\{-6B_3+42(B_1+1)B_2-56(B_1+1)^3\}\frac{4!\,A_4}{\Sigma_1^{\,6}}$$
$$+2\{B_3-6(2B_1+1)B_2+21B_1(B_1+1)^2\}\frac{5!\,A_5}{\Sigma_1^{\,5}}$$
$$+\frac{2^2}{2!}\{2B_1B_2-6B_1^{\,2}(B_1+1)\}\frac{6!\,A_6}{\Sigma_1^{\,4}}+\frac{2^3}{3!}B_1^{\,3}\frac{7!\,A_7}{\Sigma_1^{\,3}}\Big]$$
$$+\Big[\{-5B_3+30(B_1+1)B_2-35(B_1+1)^3\}\frac{5!\,A_5}{\Sigma_1^{\,5}}$$
$$+2\{B_3-5(2B_1+1)B_2+15B_1(B_1+1)^2\}\frac{6!\,A_6}{\Sigma_1^{\,4}}$$
$$+\frac{2^2}{2!}\{2B_1B_2-5B_1^{\,2}(B_1+1)\}\frac{7!\,A_7}{\Sigma_1^{\,3}}+\frac{2^3}{3!}B_1^{\,3}\frac{8!\,A_8}{\Sigma_1^{\,2}}\Big]$$
$$=-10\frac{H}{a}\left(\frac{\sigma_0}{\Sigma_1}\right)^{10}\{B_3-\tfrac{11}{3}(3B_1+1)B_2+22(B_1+1)^2(B_1-1)\}.$$

The equation for A_9 is

$$105\Bigg[\Bigg\{-9B_4+9\,.\,10\,(B_1+1)\,B_3+\frac{9\,.\,10}{2\,!}\,B_2^2-3\,\frac{9\,.\,10\,.\,11}{3\,!}\,(B_1+1)^2\,B_2$$

$$+\frac{9\,.\,10\,.\,11\,.\,12}{4\,!}\,(B_1+1)^4\Bigg\}\frac{A_1}{\Sigma_1^{\,9}}$$

$$+2\Bigg\{B_4-9\,(2B_1+1)\,B_3-9B_2^2+\frac{9\,.\,10}{2\,!}(3B_1+1)\,(B_1+1)\,B_2-\frac{9\,.\,10\,.\,11}{3\,!}B_1\,(B_1+1)^3\Bigg\}\frac{2\,!\,A_2}{\Sigma_1^{\,8}}$$

$$+\frac{2^2}{2\,!}\Bigg\{2B_1B_3+B_2^2-9B_1\,(3B_1+2)\,B_2+\frac{9\,.\,10}{2\,!}\,B_1^2\,(B_1+1)^2\Bigg\}\frac{3\,!\,A_3}{\Sigma_1^{\,7}}$$

$$+\frac{2^3}{3\,!}\,\{3B_1^2B_2-9B_1^3\,(B_1+1)\}\,\frac{4\,!\,A_4}{\Sigma_1^{\,6}}+\frac{2^4}{4\,!}\,B_1^4\frac{5\,!\,A_5}{\Sigma_1^{\,5}}\Bigg]$$

$$-105\,[\ldots]+45\,[\ldots]-10\,[\ldots]$$

$$+\Bigg[\Bigg\{-5B_4+5\,.\,6\,(B_1+1)\,B_3+\frac{5\,.\,6}{2\,!}B_2^2-3\,\frac{5\,.\,6\,.\,7}{3\,!}\,(B_1+1)^2\,B_2+\frac{5\,.\,6\,.\,7\,.\,8}{4\,!}(B_1+1)^4\Bigg\}\frac{5\,!\,A_5}{\Sigma_1^{\,5}}$$

$$+2\Bigg\{B_4-5\,(2B_1+1)\,B_3-5B_2^2+\frac{5\,.\,6}{2\,!}\,(3B_1+1)\,(B_1+1)\,B_2-\frac{5\,.\,6\,.\,7}{3\,!}\,B_1\,(B_1+1)^3\Bigg\}\frac{6\,!\,A_6}{\Sigma_1^{\,4}}$$

$$+\frac{2^2}{2\,!}\Bigg\{2B_1B_3+B_2^2-5B_1\,(3B_1+2)\,B_2+\frac{5\,.\,6}{2\,!}\,B_1^2\,(B_1+1)^2\Bigg\}\frac{7\,!\,A_7}{\Sigma_1^{\,3}}$$

$$+\frac{2^3}{3\,!}\,\{3B_1^2B_2-5B_1^3\,(B_1+1)\}\,\frac{8\,!\,A_8}{\Sigma_1^{\,2}}+\frac{2^4}{4\,!}\,B_1^4\,\frac{9\,!\,A_9}{\Sigma_1}\Bigg]$$

$$=-10\,\frac{H}{\alpha}\left(\frac{\sigma_0}{\Sigma_1}\right)^{10}\Bigg[B_4-\frac{11}{2}\,(2B_1+1)\,B_3-\frac{11}{2}\,B_2^2+\frac{11\,.\,12}{2\,!}\,B_1\,(B_1+1)\,B_2$$

$$-\frac{11\,.\,12\,.\,13}{3\,!}\,\frac{(B_1+1)^3}{4}\,(B_1-1)\Bigg],$$

where the law of formation of the terms that are not written down is sufficiently obvious.

The formulæ of this article may, of course, be transformed into those of the previous article by means of the relations by which the coefficients B were expressed in terms of the differential coefficients of Q_1, and the relations by which the coefficients $A_1, \ldots, A_5$ were expressed in terms of the same differential coefficients. They are useful in numerical work as affording a verification of the values obtained for the coefficients $A_6, A_7, \ldots$, from the previous formulæ.

Formulæ similar to those of the present and preceding articles may be obtained for the coefficients in the expression for Z belonging to the first reflected wave from the right, but it seems hardly worth while to write them down.

THE SECOND MIDDLE WAVE.

26. *Method of determining the Second Middle Wave.*—The first reflected wave from the left meets that from the right at the place and time determined by substituting

R_1 for r and s_1 for s in the formulæ belonging to the first middle wave. When they meet, the first middle wave becomes obliterated, and the second middle wave begins to be generated at the time and place in question, and encroaches upon the two first reflected waves.

To determine the second middle wave we have the conditions that at its advancing front, where $r = R_1$, the Z belonging to it is equal to that belonging to the first reflected wave from the right, and at its receding front, where $s = s_1$, the Z belonging to it is equal to that belonging to the first reflected wave from the left. RIEMANN's method may be applied in exactly the same way as in Article 15. If P is the point (r', s'), A the point (R_1, s'), B the point (R_1, s_1) and C the point (r', s_1), we have

$$Z(r', s') = [VZ]_A - [VZ]_B + [VZ]_C + \int_{AB} Z\left(\frac{\partial V}{\partial s} - \frac{5V}{r+s}\right) ds - \int_{BC} Z\left(\frac{\partial V}{\partial r} - \frac{5V}{r+s}\right) dr.$$

At A we have

$$r = R_1, \qquad s = s', \qquad \zeta = 0, \qquad V = \left(\frac{R_1+s'}{r'+s'}\right)^5,$$

and Z is the result of substituting R_1 for r and s' for s in the formula

$$Z = k_1 + l_1 u + \left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4 \left[\frac{1}{\sigma}\left\{\frac{c\sigma_0^{10}}{945a} + f_1(\sigma - u)\right\}\right].$$

At B we have

$$r = R_1, \quad s = s_1, \quad \zeta = -\frac{(R_1 - r')(s_1 - s')}{(r'+s')(R_1+s_1)}, \quad V = \left(\frac{R_1+s_1}{r'+s'}\right)^5 (1 - 20\zeta + 90\zeta^2 - 140\zeta^3 + 70\zeta^4),$$

and Z is the result of substituting R_1 for r and s_1 for s in the formulæ for the first middle wave, or in those for either of the first reflected waves. For the present we shall denote it by Z_B, and observe that it is independent of r' and s'.

At C we have

$$r = r', \qquad s = s_1, \qquad \zeta = 0, \qquad V = \left(\frac{r'+s_1}{r'+s'}\right)^5,$$

and Z is the result of substituting r' for r and s_1 for s in the formula

$$Z = K_1 + L_1 u + \left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4 \left\{\frac{F_1(\sigma+u)}{\sigma}\right\}.$$

Along AB, where $r = R_1$ and s increases from s' to s_1, we have

$$Z = k_1 + l_1(R_1 - s) + \left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4 \left[\frac{1}{\sigma}\left\{\frac{c\sigma_0^{10}}{945a} + f_1(\sigma - u)\right\}\right],$$

$$\frac{\partial V}{\partial s} - \frac{5V}{r+s} = \frac{(R_1+s')(R_1-r')(R_1+s)^3}{(r'+s')^6}(20 - 180\zeta + 420\zeta^2 - 280\zeta^3),$$

$$\zeta = -\frac{(R_1-r')(s-s')}{(r'+s')(R_1+s)};$$

and along BC, where $s = s_1$ and r decreases from R_1 to r', we have

$$Z = K_1 + L_1(r - s_1) + \left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4 \left\{\frac{F_1(\sigma + u)}{\sigma}\right\},$$

$$\frac{\partial V}{\partial r} - \frac{5V}{r+s} = \frac{(s_1+r')(s_1-s')(s_1+r)^3}{(r'+s')^6}(20 - 180\zeta + 420\zeta^2 - 280\zeta^3),$$

$$\zeta = -\frac{(r-r')(s_1-s')}{(r'+s')(r+s_1)}.$$

The value of Z at (r', s') can be regarded as a sum of terms with the coefficients

$$Z_B,\ k_1,\ l_1,\ c\sigma_0^{10}/945a,\ a_0,\ a_1, \ldots,\ K_1,\ L_1,\ A_0,\ A_1, \ldots,$$

and each of these terms may be found from the formula for $Z(r', s')$ by performing the integrations where necessary. The result will be to exhibit $Z(r', s')$ as a sum of terms with these coefficients.

27. *Determination of the Second Middle Wave.*—No integration is needed in order to obtain the term which has Z_B as a factor, but it is important to observe that V_B, as a function of r' and s', can be expressed either in the form

$$V_B = \tfrac{2}{3}(R_1 + s_1)\left(\frac{1}{\sigma'}\frac{\partial}{\partial\sigma'}\right)^4 \left\{\frac{(s'-s_1)^4(s'+R_1)^4}{\sigma'}\right\},$$

or in the form

$$V_B = \tfrac{2}{3}(R_1 + s_1)\left(\frac{1}{\sigma'}\frac{\partial}{\partial\sigma'}\right)^4 \left\{\frac{(r'-R_1)^4(r'+s_1)^4}{\sigma'}\right\}.$$

We shall suppress the accents on r' and s' so as to express the value of Z at (r, s). The term with coefficient Z_B is

$$-\tfrac{2}{3}(R_1 + s_1) Z_B \left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4 \left\{\frac{(s-s_1)^4(s+R_1)^4}{\sigma}\right\}.$$

The term with coefficient k_1 is

$$\tfrac{2}{3}k_1 \left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4 \left\{\frac{(s-s_1)^4(s+R_1)^5}{\sigma}\right\}.$$

The term with coefficient l_1 is

$$\tfrac{2}{15}l_1 \left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4 \left\{\frac{(s-s_1)^4(5R_1 - 4s_1 - s)(s+R_1)^5}{\sigma}\right\}.$$

The terms with coefficients $c\sigma_0^{10}/945a$ and a_0 are

$$\left(c\frac{\sigma_0^{10}}{945a} + a_0\right)\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4 \left[\frac{1}{\sigma}\left(\frac{s-s_1}{R_1+s_1}\right)^4 \left\{1 + 4\left(\frac{s+R_1}{R_1+s_1}\right) + 10\left(\frac{s+R_1}{R_1+s_1}\right)^2 + 20\left(\frac{s+R_1}{R_1+s_1}\right)^3 + 35\left(\frac{s+R_1}{R_1+s_1}\right)^4\right\}\right].$$

The terms with coefficients a_1, a_2, a_3 are

$$-2a_1(\mathrm{R}_1+s_1)\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left[\frac{1}{\sigma}\left(\frac{s-s_1}{\mathrm{R}_1+s_1}\right)^4\left\{1+3\left(\frac{s+\mathrm{R}_1}{\mathrm{R}_1+s_1}\right)+6\left(\frac{s+\mathrm{R}_1}{\mathrm{R}_1+s_1}\right)^2+10\left(\frac{s+\mathrm{R}_1}{\mathrm{R}_1+s_1}\right)^3+15\left(\frac{s+\mathrm{R}_1}{\mathrm{R}_1+s_1}\right)^4\right\}\right],$$

$$2^2a_2(\mathrm{R}_1+s_1)^2\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left[\frac{1}{\sigma}\left(\frac{s-s_1}{\mathrm{R}_1+s_1}\right)^4\left\{1+2\left(\frac{s+\mathrm{R}_1}{\mathrm{R}_1+s_1}\right)+3\left(\frac{s+\mathrm{R}_1}{\mathrm{R}_1+s_1}\right)^2+4\left(\frac{s+\mathrm{R}_1}{\mathrm{R}_1+s_1}\right)^3+5\left(\frac{s+\mathrm{R}_1}{\mathrm{R}_1+s_1}\right)^4\right\}\right],$$

$$-2^3a_3(\mathrm{R}_1+s_1)^3\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left[\frac{1}{\sigma}\left(\frac{s-s_1}{\mathrm{R}_1+s_1}\right)^4\left\{1+\left(\frac{s+\mathrm{R}_1}{\mathrm{R}_1+s_1}\right)+\left(\frac{s+\mathrm{R}_1}{\mathrm{R}_1+s_1}\right)^2+\left(\frac{s+\mathrm{R}_1}{\mathrm{R}_1+s_1}\right)^3+\left(\frac{s+\mathrm{R}_1}{\mathrm{R}_1+s_1}\right)^4\right\}\right].$$

The terms with coefficients a_4, a_5, ..., are

$$2^4a_4\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{(s-s_1)^4}{\sigma}\right\},\qquad 2^5a_5\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{(s-s_1)^5}{\sigma}\right\},$$

and so on.

The term with coefficient K_1 is

$$\tfrac{2}{3}\mathrm{K}_1\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{(r-\mathrm{R}_1)^4(r+s_1)^5}{\sigma}\right\}.$$

The term with coefficient L_1 is

$$-\tfrac{2}{15}\mathrm{L}_1\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{(r-\mathrm{R}_1)^4(5s_1-4\mathrm{R}_1-r)(r+s_1)^5}{\sigma}\right\}.$$

The terms with coefficients A_0, A_1, A_2, A_3 are

$$\mathrm{A}_0\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left[\frac{1}{\sigma}\left(\frac{r-\mathrm{R}_1}{\mathrm{R}_1+s_1}\right)^4\left\{1+4\left(\frac{r+s_1}{\mathrm{R}_1+s_1}\right)+10\left(\frac{r+s_1}{\mathrm{R}_1+s_1}\right)^2+20\left(\frac{r+s_1}{\mathrm{R}_1+s_1}\right)^3+35\left(\frac{r+s_1}{\mathrm{R}_1+s_1}\right)^4\right\}\right],$$

$$-2\mathrm{A}_1(\mathrm{R}_1+s_1)\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left[\frac{1}{\sigma}\left(\frac{r-\mathrm{R}_1}{\mathrm{R}_1+s_1}\right)^4\left\{1+3\left(\frac{r+s_1}{\mathrm{R}_1+s_1}\right)+6\left(\frac{r+s_1}{\mathrm{R}_1+s_1}\right)^2+10\left(\frac{r+s_1}{\mathrm{R}_1+s_1}\right)^3+15\left(\frac{r+s_1}{\mathrm{R}_1+s_1}\right)^4\right\}\right],$$

$$2^2\mathrm{A}_2(\mathrm{R}_1+s_1)^2\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left[\frac{1}{\sigma}\left(\frac{r-\mathrm{R}_1}{\mathrm{R}_1+s_1}\right)^4\left\{1+2\left(\frac{r+s_1}{\mathrm{R}_1+s_1}\right)+3\left(\frac{r+s_1}{\mathrm{R}_1+s_1}\right)^2+4\left(\frac{r+s_1}{\mathrm{R}_1+s_1}\right)^3+5\left(\frac{r+s_1}{\mathrm{R}_1+s_1}\right)^4\right\}\right],$$

$$-2^3\mathrm{A}_3(\mathrm{R}_1+s_1)^3\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left[\frac{1}{\sigma}\left(\frac{r-\mathrm{R}_1}{\mathrm{R}_1+s_1}\right)^4\left\{1+\left(\frac{r+s_1}{\mathrm{R}_1+s_1}\right)+\left(\frac{r+s_1}{\mathrm{R}_1+s_1}\right)^2+\left(\frac{r+s_1}{\mathrm{R}_1+s_1}\right)^3+\left(\frac{r+s_1}{\mathrm{R}_1+s_1}\right)^4\right\}\right].$$

The terms with coefficients A_4, A_5, ..., are

$$2^4\mathrm{A}_4\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{(r-\mathrm{R}_1)^4}{\sigma}\right\},\qquad 2^5\mathrm{A}_5\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{(r-\mathrm{R}_1)^5}{\sigma}\right\},$$

and so on.

28. *Indication of a General Method.*—If the coefficients a or A with suffixes exceeding 10 were all zero, the expression for Z in the second middle wave could be transformed at once to the form

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{Q_2(\sigma+u)}{\sigma}\right\}+K_2+L_2u,$$

or to the form

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{q_2(\sigma-u)}{\sigma}\right\}+k_2+l_2u.$$

But if these coefficients do not vanish a transformation of the same kind is still possible. We have, for example,

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{(\sigma+u)^{11}+(\sigma-u)^{11}}{\sigma}\right\}=2^5 5!\,(\sigma^2+11u^2),$$

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{(\sigma+u)^{12}-(\sigma-u)^{12}}{\sigma}\right\}=2^7 5!\,u\,(3\sigma^2+11u^2),$$

and thus Z can be expressed either in the form

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{Q_2(\sigma+u)}{\sigma}\right\}+K_2+L_2u+M_2(\sigma^2+11u^2)+N_2u\,(3\sigma^2+11u^2)+\ldots,$$

or in the form

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{q_2(\sigma-u)}{\sigma}\right\}+k_2+l_2u+m_2(\sigma^2+11u^2)+n_2u\,(3\sigma^2+11u^2)+\ldots,$$

where the factors whose coefficients are written K_2, L_2, ... , or k_2, l_2, ... , are the homogeneous rational integral functions of σ and u of degrees 0, 1, 2, 3, ... , which satisfy the differential equation for Z.

When this transformation is effected we may proceed to determine the second reflected waves. The first step is to find sets of coefficients analogous to A_0, ... , A_5 and a_0, ... , a_5. The next is to find sets of coefficients analogous to B_1, B_2, ... , determining the loci in the plane of (r, s) along which $x_0 = 0$ and $x_0 = c$ during the time that these reflected waves are being generated. By means of the coefficients analogous to B_1, B_2, ... , sets of coefficients analogous to A_6, A_7, ... , and a_6, a_7, ... , may be found, and thus the second reflected waves may be determined.

From the formulæ for Z in the second reflected waves that in the third middle wave may be found, in the same way as the formula for Z in the second middle wave was found from those in the first reflected waves.

The method of solution can be continued, and gives a theoretically complete solution of the problem; but when arithmetical computation is attempted, failure may arise through approximate equality of groups of terms with opposite signs, so that some quantity, which ought to be calculable to five figures, for example, may only be calcu-

lable to three. This difficulty was found to present itself in the calculation of the second reflected waves by this method, and another method had to be sought. An account of this will be given in the theory of the second reflected waves.

29. *Pistons of Equal Mass.*—A considerable reduction in the number of coefficients to be calculated is effected by supposing the two pistons to have the same mass. When this is so $\mathrm{H} = h$, and hereafter we shall write everywhere h for H. The calculation of $\mathrm{A}_0, \mathrm{A}_1, \ldots, \mathrm{A}_5$ is then simplified a good deal. Further, it appears that the coefficients a differ only in sign from the coefficients A, or we have

$$a_0 = -\mathrm{A}_0, \qquad a_1 = -\mathrm{A}_1, \ldots.$$

It is now unnecessary to calculate separately the pressures, velocities, displacements, and times at the two pistons. We shall speak of the piston specified by $x_0 = c$ as the "shot," and of the piston specified by $x_0 = 0$ as the "image of the shot." We shall generally calculate the pressures, &c., for the image of the shot, because a slight simplification is effected by putting x_0 equal to zero.

30. *Incidence of the Second Middle Wave upon the Pistons.*—The value of s at the receding front of the second middle wave is that which has been denoted by s_1, and in the case of equal pistons it is the same as R_1 or $\Sigma_1 - \frac{1}{2}\sigma_0$. This is therefore the value of s at the image of the shot at the instant when the receding front of the second middle wave reaches it. It will be denoted by S_2. The corresponding value of r may be found from the formula

$$r - \mathrm{R}_1 = \mathrm{B}_1(s - \mathrm{S}_1) + (\mathrm{B}_2/\Sigma_1)(s - \mathrm{S}_1)^2 + \ldots$$

by putting S_2 for s. It will be denoted by R_2. From this the corresponding value of σ may be found. It will be denoted by Σ_2. The corresponding value of u, which is $\mathrm{R}_2 - \mathrm{S}_2$, will be denoted by U_2. The corresponding value of Z, denoted by Z_2, can be found most simply from the formula for the first reflected wave from the left. We have

$$\mathrm{Z}_2 = \mathrm{K}_1 + \mathrm{L}_1\mathrm{U}_2 + 105\,\frac{\mathrm{F}_1(2\mathrm{R}_2)}{\Sigma_2^{\,9}} - 105\,\frac{\mathrm{F}_1^{(1)}(2\mathrm{R}_2)}{\Sigma_2^{\,8}} + 45\,\frac{\mathrm{F}_1^{(2)}(2\mathrm{R}_2)}{\Sigma_2^{\,7}} - 10\,\frac{\mathrm{F}_1^{(3)}(2\mathrm{R}_2)}{\Sigma_2^{\,6}} + \frac{\mathrm{F}_1^{(4)}(2\mathrm{R}_2)}{\Sigma_2^{\,5}},$$

where

$$\frac{\mathrm{F}_1(2\mathrm{R}_2)}{\Sigma_2^{\,10}} = \left(\frac{\Sigma_1}{\Sigma_2}\right)^{10}\left\{\frac{\mathrm{A}_0}{\Sigma_1^{\,10}} - \frac{\mathrm{A}_1}{\Sigma_1^{\,9}}\left(\frac{2\mathrm{R}_1 - 2\mathrm{R}_2}{\Sigma_1}\right) + \frac{1}{2!}\,\frac{2!\,\mathrm{A}_2}{\Sigma_1^{\,8}}\left(\frac{2\mathrm{R}_1 - 2\mathrm{R}_2}{\Sigma_1}\right)^2 - \frac{1}{3!}\,\frac{3!\,\mathrm{A}_3}{\Sigma_1^{\,7}}\left(\frac{2\mathrm{R}_1 - 2\mathrm{R}_2}{\Sigma_1}\right)^3 + \ldots\right\},$$

$$\frac{\mathrm{F}_1^{(1)}(2\mathrm{R}_2)}{\Sigma_2^{\,9}} = \left(\frac{\Sigma_1}{\Sigma_2}\right)^{9}\left\{\frac{\mathrm{A}_1}{\Sigma_1^{\,9}} - \frac{2!\,\mathrm{A}_2}{\Sigma_1^{\,8}}\left(\frac{2\mathrm{R}_1 - 2\mathrm{R}_2}{\Sigma_1}\right) + \frac{1}{2!}\,\frac{3!\,\mathrm{A}_3}{\Sigma_1^{\,7}}\left(\frac{2\mathrm{R}_1 - 2\mathrm{R}_2}{\Sigma_1}\right)^2 - \frac{1}{3!}\,\frac{4!\,\mathrm{A}_4}{\Sigma_1^{\,6}}\left(\frac{2\mathrm{R}_1 - 2\mathrm{R}_2}{\Sigma_1}\right)^3 + \ldots\right\},$$

$$\frac{F_1^{(2)}(2R_2)}{\Sigma_2^{\,8}} = \left(\frac{\Sigma_1}{\Sigma_2}\right)^8 \left\{\frac{2!\,A_2}{\Sigma_1^{\,8}} - \frac{3!\,A_3}{\Sigma_1^{\,7}}\left(\frac{2R_1-2R_2}{\Sigma_1}\right) + \frac{1}{2!}\frac{4!\,A_4}{\Sigma_1^{\,6}}\left(\frac{2R_1-2R_2}{\Sigma_1}\right)^2 - \frac{1}{3!}\frac{5!\,A_5}{\Sigma_1^{\,5}}\left(\frac{2R_1-2R_2}{\Sigma_1}\right)^3 + \dots\right\},$$

$$\frac{F_1^{(3)}(2R_2)}{\Sigma_2^{\,7}} = \left(\frac{\Sigma_1}{\Sigma_2}\right)^7 \left\{\frac{3!\,A_3}{\Sigma_1^{\,7}} - \frac{4!\,A_4}{\Sigma_1^{\,6}}\left(\frac{2R_1-2R_2}{\Sigma_1}\right) + \frac{1}{2!}\frac{5!\,A_5}{\Sigma_1^{\,5}}\left(\frac{2R_1-2R_2}{\Sigma_1}\right)^2 - \frac{1}{3!}\frac{6!\,A_6}{\Sigma_1^{\,4}}\left(\frac{2R_1-2R_2}{\Sigma_1}\right)^3 + \dots\right\},$$

$$\frac{F_1^{(4)}(2R_2)}{\Sigma_2^{\,6}} = \left(\frac{\Sigma_1}{\Sigma_2}\right)^6 \left\{\frac{4!\,A_4}{\Sigma_1^{\,6}} - \frac{5!\,A_5}{\Sigma_1^{\,5}}\left(\frac{2R_1-2R_2}{\Sigma_1}\right) + \frac{1}{2!}\frac{6!\,A_6}{\Sigma_1^{\,4}}\left(\frac{2R_1-2R_2}{\Sigma_1}\right)^2 - \frac{1}{3!}\frac{7!\,A_7}{\Sigma_1^{\,3}}\left(\frac{2R_1-2R_2}{\Sigma_1}\right)^3 + \dots\right\}.$$

31. *Transformation of the Formula for the Second Middle Wave.*—In what follows we shall disregard coefficients A beyond A_9; if it were desired to include further coefficients A some of the formulæ would require modification, but there is no difficulty arising from the convention to stop at A_9. The most effective transformation of the formula for Z in the second middle wave is found by putting for Z_B the value derived from the first reflected wave from the right, viz. :—

$$Z_B = k_1 + l_1(R_1 - s_1) + \left[\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4 \left\{\frac{c\sigma_0^{\,10}}{945a\sigma} + \frac{a_0 + a_1(2s-2s_1) + a_2(2s-2s_1)^2 + \dots}{\sigma}\right\}\right]_{r=R_1,\, s=S_1},$$

so that the terms contributed to Z by Z_B come to

$$-\tfrac{2}{3}\Big\{k_1(R_1+s_1)^9 + 105\left(\frac{c\sigma_0^{\,10}}{945a} + a_0\right) - 105(R_1+s_1)a_1 + 45(R_1+s_1)^2\, 2!\, a_2 - 10(R_1+s_1)^3\, 3!\, a_3 + (R_1+s_1)^4\, 4!\, a_4\Big\} \times \left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4 \left\{\frac{(s-s_1)^4(s+R_1)^4}{\sigma(R_1+s_1)^8}\right\},$$

and then, before putting R_1 for s_1, or $-A_0$, $-A_1$, ..., for a_0, a_1, ..., transforming the terms contributed by A_0, A_1, A_2, A_3 to the form

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4 \left[\frac{1}{\sigma}\left\{A_0 + 2A_1(r-R_1) + 2^2A_2(r-R_1)^2 + 2^3A_3(r-R_1)^3\right\}\right]$$

$$\begin{aligned}
+ \left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4 \Bigg[&\frac{A_0}{\sigma}\left\{-56\left(\frac{s_1+r}{R_1+s_1}\right)^5 + 140\left(\frac{s_1+r}{R_1+s_1}\right)^6 - 120\left(\frac{s_1+r}{R_1+s_1}\right)^7 + 35\left(\frac{s_1+r}{R_1+s_1}\right)^8\right\} \\
&- \frac{2A_1(R_1+s_1)}{\sigma}\left\{-21\left(\frac{s_1+r}{R_1+s_1}\right)^5 + 56\left(\frac{s_1+r}{R_1+s_1}\right)^6 - 50\left(\frac{s_1+r}{R_1+s_1}\right)^7 + 15\left(\frac{s_1+r}{R_1+s_1}\right)^8\right\} \\
&+ \frac{2^2A_2(R_1+s_1)^2}{\sigma}\left\{-6\left(\frac{s_1+r}{R_1+s_1}\right)^5 + 17\left(\frac{s_1+r}{R_1+s_1}\right)^6 - 16\left(\frac{s_1+r}{R_1+s_1}\right)^7 + 5\left(\frac{s_1+r}{R_1+s_1}\right)^8\right\} \\
&- \frac{2^3A_3(R_1+s_1)^3}{\sigma}\left\{-\left(\frac{s_1+r}{R_1+s_1}\right)^5 + 3\left(\frac{s_1+r}{R_1+\ _1}\right)^6 - 3\left(\frac{s_1+r}{R_1+s_1}\right)^7 + \left(\frac{s_1+r}{R_1+s_1}\right)^8\right\}\Bigg].
\end{aligned}$$

The first line of this expression, with the terms contributed by A_4, A_5, ..., makes up

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{F_1(\sigma+u)}{\sigma}\right\},$$

and the remaining lines are unaltered when $-s$ is written for r.

The terms contributed by K_1 and L_1 are the same as

$$K_1+L_1u-\tfrac{2}{3}K_1\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{(s-s_1)^5(R_1+s)^4}{\sigma}\right\}+\tfrac{2}{15}L_1\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{(s-s_1)^5(R_1+s)^5}{\sigma}\right\},$$

and thus the Z of the second middle wave is expressed entirely as the sum of the Z of the first reflected wave from the left and a function of the form

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{\phi_2(\sigma-u)}{\sigma}\right\}.$$

Further, noting that with equal pistons $l_1 = L_1$, we see that $\phi_2(\sigma-u)$ contains no terms of degree higher than the ninth in s or $\frac{1}{2}(\sigma-u)$. Also we see that it can be expressed as a rational integral function of $(s-s_1)/(R_1+s_1)$ of the ninth degree, and that it contains no terms of degree lower than the fourth. Since Z and $\partial Z/\partial\sigma$ are continuous at $s = s_1$ with the Z and $\partial Z/\partial\sigma$ belonging to the first reflected wave from the left, the function ϕ_2 can contain no terms of the fourth or fifth degree in $(s-s_1)/(R_1+s_1)$. The vanishing of the coefficients of these terms does not introduce any new condition. On replacing a_0, ... by $-A_0$, ..., we have the result that in the second middle wave

$$Z = K_1+L_1u+\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{F_1(\sigma+u)}{\sigma}\right\}+\Sigma_2^{10}\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{1}{\sigma}\left(\eta_6\xi^6-\eta_7\xi^7+\eta_8\xi^8-\eta_9\xi^9\right)\right\},$$

where ξ is written for $(s-s_1)/(R_1+s_1)$, and $s_1 = R_1 = S_2$, while η_6, η_7, η_8, η_9 are given by the equations

$$\eta_6 = -\tfrac{8}{3}(K_1-k_1)\frac{(2R_1)^9}{\Sigma_2^{10}}-\frac{4c}{27a}\left(\frac{\sigma_0}{\Sigma_2}\right)^{10}+280\zeta_0+140\zeta_1+62\zeta_2+23\zeta_3+6\zeta_4-\zeta_6,$$

$$\eta_7 = 4(K_1-k_1)\frac{(2R_1)^9}{\Sigma_2^{10}}-\frac{8c}{63a}\left(\frac{\sigma_0}{\Sigma_2}\right)^{10}-240\zeta_0-120\zeta_1-52\zeta_2-18\zeta_3-4\zeta_4-\zeta_7,$$

$$\eta_8 = -\tfrac{8}{3}(K_1-k_1)\frac{(2R_1)}{\Sigma_2^{10}}-\frac{c}{27a}\left(\frac{\sigma_0}{\Sigma_2}\right)^{10}+70\zeta_0+35\zeta_1+15\zeta_2+5\zeta_3+\zeta_4-\zeta_8,$$

$$\eta_9 = \tfrac{2}{3}(K_1-k_1)\frac{(2R_1)^9}{\Sigma_2^{10}}-\zeta_9,$$

in which

$$\zeta_0 = \frac{A_0}{\Sigma_1^{10}}\left(\frac{\Sigma_1}{\Sigma_2}\right)^{10}, \quad \zeta_2 = -2\frac{A_1}{\Sigma_1^{9}}\frac{2R_1}{\Sigma_1}\left(\frac{\Sigma_1}{\Sigma_2}\right)^{10}, \quad \zeta_3 = 2\frac{2!\,A_2}{\Sigma_1^{8}}\left(\frac{2R_1}{\Sigma_1}\right)^{2}\left(\frac{\Sigma_1}{\Sigma_2}\right)^{10},$$

$$\zeta_3 = -\tfrac{4}{3}\frac{3!\,A_3}{\Sigma_1^{7}}\left(\frac{2R_1}{\Sigma_1}\right)^{3}\left(\frac{\Sigma_1}{\Sigma_2}\right)^{10}, \quad \zeta_4 = \tfrac{2}{3}\frac{4!\,A_4}{\Sigma_1^{6}}\left(\frac{2R_1}{\Sigma_1}\right)^{4}\left(\frac{\Sigma_1}{\Sigma_2}\right)^{10}, \quad \zeta_5 = -\tfrac{4}{15}\frac{5!\,A_5}{\Sigma_1^{5}}\left(\frac{2R_1}{\Sigma_1}\right)^{5}\left(\frac{\Sigma_1}{\Sigma_2}\right)^{10},$$

$$\zeta_6 = \tfrac{4}{45}\frac{6!\,A_6}{\Sigma_1^{4}}\left(\frac{2R_1}{\Sigma_1}\right)^{6}\left(\frac{\Sigma_1}{\Sigma_2}\right)^{10}, \quad \zeta_7 = -\tfrac{8}{315}\frac{7!\,A_7}{\Sigma_1^{3}}\left(\frac{2R_1}{\Sigma_1}\right)^{7}\left(\frac{\Sigma_1}{\Sigma_2}\right)^{10}, \quad \zeta_8 = \tfrac{2}{315}\frac{8!\,A_8}{\Sigma_1^{2}}\left(\frac{2R_1}{\Sigma_1}\right)^{8}\left(\frac{\Sigma_1}{\Sigma_2}\right)^{10},$$

$$\zeta_9 = -\tfrac{4}{2835}\frac{9!\,A_9}{\Sigma_1}\left(\frac{2R_1}{\Sigma_1}\right)^{9}\left(\frac{\Sigma_1}{\Sigma_2}\right)^{10}.$$

The Second Reflected Waves.

32. *Relation between Pressure and Velocity at a Piston.*—The relation between pressure and velocity at the image of the shot is an equation connecting r and s, which holds at $x_0 = 0$, and can be interpreted as the equation of a certain locus in the plane of r and s. This equation can be written in the form

$$(r - R_2)/\Sigma_2 = B'_1\delta + B'_2\delta^2 + B'_3\delta^3 + \dots,$$

where δ stands for $(s - S_2)/\Sigma_2$, and the coefficients B' are at present undetermined.

To determine these coefficients we have recourse to the method of Articles 21–23. During the progress of the second reflected wave from the left, the value of x_0 at any point in the region occupied by it can be expressed in terms of the values r' and s' of r and s, which occur simultaneously at the point, by the formula

$$x_0 = -5h\int_{AC} (Y/\sigma)\, du,$$

wherein the integral is taken along the locus from the point A, where $r = r'$, to the point C, where $s = s'$. In this integral

$$u - U_2 = \Sigma_2\{(B'_1 - 1)\delta + B'_2\delta^2 + B'_3\delta^3 + \dots\},$$

$$du = \Sigma_2\{(B'_1 - 1) + 2B'_2\delta + 3B'_3\delta^2 + \dots\}\, d\delta,$$

$$\sigma = \Sigma_2\{1 + (B'_1 + 1)\delta + B'_2\delta^2 + B'_3\delta^3 + \dots\},$$

$$\frac{Y}{\sigma} = \frac{(r' + s')^5}{\sigma^6}\left\{1 + 30\frac{(r - r')(s - s')}{(r' + s')\sigma} + 210\frac{(r - r')^2(s - s')^2}{(r' + s')^2\sigma^2} + \dots\right\}.$$

On putting R_2 for r', we have the value of x_0 along the junction of the second middle wave and the second reflected wave from the left expanded in a series of powers of δ', or $(s'-S_2)/\Sigma_2$, in the form

$$x_0 = -5h\,[(B'_1-1)\,\delta'+\{B'_2-(3B'_1-2)\,(B'_1-1)\}\,\delta'^2$$
$$+\{B'_3-3B'_2\,(2B'_1-1)+(7B'^2_1-6B'_1+2)\,(B'_1-1)\}\,\delta'^3$$
$$+\{B'_4-2B'_3\,(3B'_1-1)-3B'^2_2-\tfrac{1}{2}B'_2\,(42B'^2_1-33B'_1+5)$$
$$-\tfrac{1}{2}\,(28B'^3_1-21B'^2_1+9B'_1-2)\,(B'_1-1)\}\,\delta'^4$$
$$+\{B'_5-\tfrac{1}{5}B'_4\,(30B'_1-7)-6B'_2B'_3+\tfrac{1}{10}B'_3\,(210B'^2_1-108B'_1+7)$$
$$+\tfrac{1}{5}B'^2_2\,(105B'_1-27)-\tfrac{1}{10}B'_2\,(560B'^3_1-462B'^2_1+81B'_1-7)$$
$$+\tfrac{1}{5}\,(126B'^4_1-56B'^3_1+21B'^2_1-6B'_1+1)\,(B'_1-1)\}\,\delta'^5+\ldots].$$

Now at any point (R_2, s') on the same junction the value of x_0 can be obtained by forming $-\Pi\,\partial Z/\partial\sigma$, where

$$Z = K_1+L_1u+\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{F_1(\sigma+u)}{\sigma}\right\}+\Sigma_2^{10}\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^4\left\{\frac{1}{\sigma}\left(\eta_6\xi^6-\eta_7\xi^7+\eta_8\xi^8-\eta_9\xi^9\right)\right\}$$

and

$$\Pi = a\,(\sigma/\sigma_0)^{10}, \qquad \xi = (s-S_2)/2R_1,$$

and putting therein

$$\sigma = R_2+s', \qquad u = R_2-s', \qquad s = s', \qquad s'-S_2 = \Sigma_2\delta',$$

and the result can be expressed in terms of δ' in the form

$$\frac{x_0}{a} = \left(\frac{\Sigma_2}{\sigma_0}\right)^{10}\left[\left\{945\,\frac{F_1(2R_2)}{\Sigma_2^{10}}-945\,(1+\delta')\,\frac{F_1^{(1)}(2R_2)}{\Sigma_2^{9}}+420\,(1+\delta')^2\,\frac{F_1^{(2)}(2R_2)}{\Sigma_2^{8}}\right.\right.$$
$$\left.-105\,(1+\delta')^3\,\frac{F_1^{(3)}(2R_2)}{\Sigma_2^{7}}+15\,(1+\delta')^4\,\frac{F_1^{(4)}(2R_2)}{\Sigma_2^{6}}-(1+\delta')^5\,\frac{F_1^{(5)}(2R_2)}{\Sigma_2^{5}}\right\}$$
$$+\eta_6\left(\frac{\Sigma_2}{2R_1}\right)^6\{945\delta'^6-945\times 3\,(1+\delta')\,\delta'^5+420\times\tfrac{15}{2}\,(1+\delta')^2\,\delta'^4$$
$$-105\times 15\,(1+\delta')^3\,\delta'^3+15\times\tfrac{45}{2}\,(1+\delta')^4\,\delta'^2-\tfrac{45}{2}\,(1+\delta')^5\,\delta'\}$$
$$-\eta_7\left(\frac{\Sigma_2}{2R_1}\right)^7\{945\delta'^7-945\times\tfrac{7}{2}\,(1+\delta')\,\delta'^6+420\times\tfrac{21}{2}\,(1+\delta')^2\,\delta'^5$$
$$-105\times\tfrac{105}{4}\,(1+\delta')^3\,\delta'^4+15\times\tfrac{105}{2}\,(1+\delta')^4\,\delta'^3-\tfrac{315}{4}\,(1+\delta')^5\,\delta'^2\}$$
$$+\eta_8\left(\frac{\Sigma_2}{2R_1}\right)^8\{945\delta'^8-945\times 4\,(1+\delta')\,\delta'^7+420\times 14\,(1+\delta')^2\,\delta'^6$$
$$-105\times 42\,(1+\delta')^3\,\delta'^5+15\times 105\,(1+\delta')^4\,\delta'^4-210\,(1+\delta')^5\,\delta'^3\}$$
$$-\eta_9\left(\frac{\Sigma_2}{2R_1}\right)^9\{945\delta'^9-945\times\tfrac{9}{2}\,(1+\delta')\,\delta'^8+420\times 18\,(1+\delta')^2\,\delta'^7$$
$$\left.-105\times 63\,(1+\delta')^3\,\delta'^6+15\times 189\,(1+\delta')^4\,\delta'^5-\tfrac{945}{2}\,(1+\delta')^5\,\delta'^4\}\right].$$

The terms independent of δ' in the right-hand member of this equation add up to zero, for x_0 vanishes at (R_2, S_2); and, by equating the coefficients of powers of δ' in the two expressions for x_0, equations are obtained from which the values of the coefficients B′ can be found successively.

33. *Relation between Velocity and Time at a Piston.*—The time at which any particular simultaneous values of r and s occur at $x_0 = 0$, can be found by the method of Article 25, and thus the relation between velocity and time at the image of the shot may be traced. We can write down the equation

$$t-T_2 = -10\frac{h}{a}\frac{\sigma_0^{10}}{\Sigma_2^{11}}\int_{S_2}^{s}\left(\frac{\sigma}{\Sigma_2}\right)^{-11}\{(B'_1-1)+2B'_2\delta+3B'_3\delta^2+\ldots\}ds,$$

in which

$$\delta = (s-S_2)/\Sigma_2, \qquad \sigma = \Sigma_2\{1+(B'_1+1)\delta+B'_2\delta^2+B'_3\delta^3+\ldots\},$$

and thus $t-T_2$ can be expanded in powers of δ in the form

$$t-T_2 = c_1\delta+c_2\delta^2+c_3\delta^3+\ldots,$$

where

$$c_1 = -\frac{10h}{a}\left(\frac{\sigma_0}{\Sigma_2}\right)^{10}(B'_1-1),$$

$$c_2 = -\frac{10h}{a}\left(\frac{\sigma_0}{\Sigma_2}\right)^{10}\{B'_2-\tfrac{11}{2}(B'_1+1)(B'_1-1)\},$$

$$c_3 = -\frac{10h}{a}\left(\frac{\sigma_0}{\Sigma_2}\right)^{10}\{B'_3-\tfrac{11}{3}(3B'_1+1)B'_2+22(B'_1+1)^2(B'_1-1)\},$$

$$c_4 = -\frac{10h}{a}\left(\frac{\sigma_0}{\Sigma_2}\right)^{10}\{B'_4-\tfrac{11}{2}(2B'_1+1)B'_3-\tfrac{11}{2}B'^2_2+66B'_1(B'_1+1)B'_2 \\ -\tfrac{143}{2}(B'_1+1)^3(B'_1-1)\},$$

$$c_5 = -\frac{10h}{a}\left(\frac{\sigma_0}{\Sigma_2}\right)^{10}\{B'_5-\tfrac{11}{5}(5B'_1+3)B'_4-11B'_2B'_3+\tfrac{132}{10}(5B'_1+1)(B'_1+1)B'_3 \\ +\tfrac{132}{10}(5B'_1+3)B'^2_2-\tfrac{572}{10}(5B'_1-1)(B'_1+1)^2B'_2 \\ +\tfrac{2002}{10}(B'_1+1)^4(B'_1-1)\},$$

. . . .

34. *Displacement of a Piston.*—To obtain the displacement of the image of the shot, we have to find the value of x at $x_0 = 0$ in terms of simultaneous values of r and s occurring on the locus

$$r-R_2 = B'_1(s-S_2)+(B'_2/\Sigma_2)(s-S_2)^2+(B'_3/\Sigma_2^2)(s-S_2)^3+\ldots.$$

Now, when $x_0 = 0$, we have $x = ut-Z$, and for the value of x at (R_2, S_2), denoted by X_2, we have $X_2 = U_2T_2-Z_2$, so that when $x_0 = 0$, we have

$$x-X_2 = ut-U_2T_2-(Z-Z_2).$$

Also we have

$$Z-Z_2 = \int_{(R_2, S_2)}^{(r, s)} \frac{\partial Z}{\partial u}\, du + \frac{\partial Z}{\partial \sigma}\, d\sigma,$$

where the integral is taken along the locus, and, since $\partial Z/\partial\sigma$ vanishes along the locus, we get

$$x-x_2 = ut-U_2T_2-\int_{(R_2, S_2)}^{(r, s)} t\,du = \int_{S_2}^{s} u\,\frac{dt}{ds}\,ds,$$

in which

$$\begin{aligned} u &= U_2+\Sigma_2\{(B'_1-1)\,\delta+B'_2\delta^2+B'_3\delta^3+\dots\}, \\ t &= T_2+c_1\delta+c_2\delta^2+c_3\delta^3+\dots, \\ s &= S_2+\Sigma_2\delta. \end{aligned}$$

It follows that, at $x_0 = 0$, x is given by the equation

$$\begin{aligned} x-X_2 = {} & U_2c_1\delta+\{U_2c_2+\tfrac{1}{2}\Sigma_2c_1(B'_1-1)\}\,\delta^2+\{U_2c_3+\tfrac{2}{3}\Sigma_2c_2(B'_1-1)+\tfrac{1}{3}\Sigma_2c_1B'_2\}\,\delta^3 \\ & +\{U_2c_4+\tfrac{3}{4}\Sigma_2c_3(B'_1-1)+\tfrac{1}{2}\Sigma_2c_2B'_2+\tfrac{1}{4}\Sigma_2c_1B'_3\}\,\delta^4 \\ & +\{U_2c_5+\tfrac{4}{5}\Sigma_2c_4(B'_1-1)+\tfrac{3}{5}\Sigma_2c_3B'_2+\tfrac{2}{5}\Sigma_2c_2B'_3+\tfrac{1}{5}\Sigma_2c_1B'_4\}\,\delta^5+\dots, \end{aligned}$$

which may be written

$$x-X_2 = \xi_1\delta+\xi_2\delta^2+\xi_3\delta^3+\dots.$$

The formulæ which have been obtained avail to determine the displacement, velocity and pressure at the shot or its image at any time during the progress of the second reflected waves.

35. *General Method for the Second Reflected Waves.*—We shall need to be able to calculate Z, t and x_0 for any simultaneous values of r and s that can occur in the second reflected wave from the left. It is best to obtain formulæ for t and x_0 separately, and not to deduce them from the formula for Z by differentiation, because the formulæ will be approximate, and to obtain the terms of any particular order in t, for example, by differentiation of Z it would be necessary to obtain the terms of order higher by one in the formula for Z. The method of determining the formula for x_0 has been indicated already in Article 32, and the work will be completed presently. The formulæ for t and Z will be found by similar applications of the method of RIEMANN. We begin with the formula for t. After finding formulæ for x_0, t, Z, we can calculate x from the equation

$$x = x_0\left\{\frac{\rho_0}{\beta}+\left(\frac{\sigma_0}{\sigma}\right)^9\left(1-\frac{\rho_0}{\beta}\right)\right\}+ut-Z.$$

36. *Method of Determining t.*—The value of t along the locus $x_0 = 0$ has been found in Article 33. To obtain the differential coefficients of t along the same locus we have the equations

$$\frac{\partial x_0}{\partial u}+\Pi\,\frac{\partial t}{\partial\sigma} = 0, \qquad \frac{\partial x_0}{\partial\sigma}+\Pi\,\frac{\partial t}{\partial u} = 0,$$

and

$$\frac{\partial x_0}{\partial \sigma} = \frac{10h}{\sigma\left\{1-\left(\frac{d\sigma}{du}\right)^2\right\}}, \qquad \frac{\partial x_0}{\partial u} = -\frac{10h\,\frac{d\sigma}{du}}{\sigma\left\{1-\left(\frac{d\sigma}{du}\right)^2\right\}},$$

where $d\sigma/du$ is to be found from the equation of the locus. Thus we may write

$$\frac{\partial t}{\partial \sigma} = 10\frac{h\sigma_0^{10}}{a\sigma^{11}}\frac{\frac{d\sigma}{du}}{1-\left(\frac{d\sigma}{du}\right)^2}, \qquad \frac{\partial t}{\partial u} = -10\frac{h\sigma_0^{10}}{a\sigma^{11}}\frac{1}{1-\left(\frac{d\sigma}{du}\right)^2}.$$

Now let (r', s') be any point P, and let lines PA, PC parallel to the axes of s and r meet the locus in A, C, as in fig. 3 in Article 22. Then, since t satisfies the same differential equation as Z, the integral

$$\int \mathrm{V}\left(\frac{\partial t}{\partial s}+5\,\frac{t}{\sigma}\right)ds + t\left(\frac{\partial \mathrm{V}}{\partial r}-5\,\frac{\mathrm{V}}{\sigma}\right)dr$$

taken round the contour formed by the arc AC and the lines CP, PA vanishes, and therefore we have the equation

$$t\,(r', s') = [\mathrm{V}t]_{\mathrm{A}} + \int_{\mathrm{AC}}\left(\frac{\partial \mathrm{V}}{\partial r}-5\,\frac{\mathrm{V}}{\sigma}\right)t\,dr + \left(\frac{\partial t}{\partial s}+5\,\frac{t}{\sigma}\right)\mathrm{V}\,ds,$$

or, on putting

$$t' = t - \mathrm{T}_2,$$

the equation

$$t'\,(r', s') = [\mathrm{V}t'] + \int_{\mathrm{AC}}\left(\frac{\partial \mathrm{V}}{\partial r}-5\,\frac{\mathrm{V}}{\sigma}\right)t'\,dr + \left(\frac{\partial t'}{\partial s}+5\,\frac{t'}{\sigma}\right)\mathrm{V}\,ds.$$

But we have, along the locus,

$$\frac{\partial t'}{\partial s} = \frac{\partial t'}{\partial \sigma} - \frac{\partial t'}{\partial u} = -10\,\frac{h\sigma_0^{10}}{a\sigma^{11}}\,\frac{1}{\frac{d\sigma}{du}-1} = -\frac{5h\sigma_0^{10}}{a\sigma^{11}}\,\frac{du}{ds},$$

and, by the theory of Article 25, the expression last written is the same as the value of $\frac{1}{2}dt/ds$ along the locus, or we have

$$\frac{\partial t'}{\partial s} = \tfrac{1}{2}\Sigma_2^{-1}\,(c_1+2c_2\delta+3c_3\delta^2+\ldots).$$

Also along the locus we have

$$t' = c_1\delta + c_2\delta^2 + c_3\delta^3 + \ldots.$$

Let s_A denote the value of s at A. Then at A we have

$$r = r', \quad s = s_A, \quad V = \left(\frac{r'+s_A}{r'+s'}\right)^5,$$

$$t' = c_1\delta_A + c_2\delta_A{}^2 + c_3\delta_A{}^3 + \ldots,$$

where δ_A stands for $(s_A - S_2)/\Sigma_2$.

Along AC we have the formulæ already written for t' and $\partial t'/\partial s$, and we have further

$$V = \left(\frac{r+s}{r'+s'}\right)^5 (1-20\zeta+90\zeta^2-140\zeta^3+70\zeta^4), \quad \zeta = -\frac{(r-r')(s-s')}{(r'+s')(r+s)},$$

$$\frac{\partial V}{\partial r} - 5\,\frac{V}{\sigma} = \frac{(s+r')(s-s')(r+s)^3}{(r'+s')^6}(20-180\zeta+420\zeta^2-280\zeta^3),$$

and we have to put

$$ds = \Sigma_2\, d\delta, \quad dr = \Sigma_2(B'_1 + 2B'_2\delta + 3B'_3\delta^2 + \ldots)\, d\delta.$$

The limits of integration are δ_A and δ', which is $(s'-S_2)/\Sigma_2$.

The value of δ_A is to be found by reversing the series

$$r - R_2 = (s-S_2)\{B'_1 + (B'_2/\Sigma_2)(s-S_2) + (B'_3/\Sigma_2{}^2)(s-S_2)^2 + \ldots\},$$

and putting r' for r and s_A for s. If we write ϵ' for $(r'-R_2)/\Sigma_2$ the result is

$$\delta_A = \frac{\epsilon'}{B'_1} - \frac{B'_2}{B'_1{}^3}\epsilon'^2 + \left\{\frac{2B'_2{}^2}{B'_1{}^5} - \frac{B'_3}{B'_1{}^4}\right\}\epsilon'^3 - \left(\frac{5B'_2{}^3}{B'_1{}^7} - \frac{5B'_2B'_3}{B'_1{}^6} + \frac{B'_4}{B'_1{}^5}\right)\epsilon'^4$$
$$+\left(14\,\frac{B'_2{}^4}{B'_1{}^9} - 21\,\frac{B'_2{}^2B'_3}{B'_1{}^8} + 3\,\frac{B'_3{}^2}{B'_1{}^7} + 6\,\frac{B'_2B'_4}{B'_1{}^7} - \frac{B'_5}{B'_1{}^6}\right)\epsilon'^5 + \ldots.$$

Thus δ_A is known in terms of r'.

37. *Formula for the Time.*—We work out the formula for t' or $t - T_2$ in terms of δ, or $(s-S_2)/\Sigma_2$, and ϵ, or $(r-R_2)/\Sigma_2$, at any point answering to simultaneous values of r and s which can occur in the second reflected wave from the left. For this we first perform the integrations with respect to r and s and then suppress the accents on δ' and ϵ'. We record the results as far as terms of the fourth order.

The terms of the first order present themselves in the form

$$\left(\frac{1+\epsilon+\delta_A}{1+\epsilon+\delta}\right)^5 c_1\delta_A + \frac{1}{(1+\epsilon+\delta)^5}\,\tfrac{1}{2}c_1(\delta-\delta_A),$$

and it is simpler to leave the factors

$$\left(\frac{1+\epsilon+\delta_A}{1+\epsilon+\delta}\right)^5, \qquad \frac{1}{(1+\epsilon+\delta)^5}$$

as they are, rather than to expand them in powers of ϵ, δ_A and δ. In like manner the terms of the second order are

$$\left(\frac{1+\epsilon+\delta_A}{1+\epsilon+\delta}\right)^5 c_2\delta_A{}^2+\frac{1}{(1+\epsilon+\delta)^5}\{\tfrac{1}{2}c_2+\tfrac{5}{4}c_1(B'_1+3)\}(\delta^2-\delta_A{}^2),$$

the terms of the third order are

$$\left(\frac{1+\epsilon+\delta_A}{1+\epsilon+\delta}\right)^5 c_3\delta_A{}^3+\frac{1}{(1+\epsilon+\delta)^5}\{\tfrac{1}{2}c_3+\tfrac{5}{3}c_2(B'_1+2)+\tfrac{5}{3}c_1(B'_1+1)(B'_1+5)+\tfrac{5}{6}c_1B'_2\}(\delta^3-\delta_A{}^3)$$
$$+\frac{5}{(1+\epsilon+\delta)^6}c_1\epsilon(\delta-\delta_A)^2-\frac{5}{(1+\epsilon+\delta)^6}c_1B'_1(\delta+2\delta_A)(\delta-\delta_A)^2,$$

and the terms of the fourth order are

$$\left(\frac{1+\epsilon+\delta_A}{1+\epsilon+\delta}\right)^5 c_4\delta_A{}^4+\frac{1}{(1+\epsilon+\delta)^5}\{\tfrac{1}{2}c_4+\tfrac{5}{8}c_3(3B'_1+5)+\tfrac{5}{2}c_2(B'_1+1)(B'_1+3)+\tfrac{5}{4}c_2B'_2$$
$$+\tfrac{5}{4}c_1(B'_1+1)^2(B'_1+7)+\tfrac{5}{2}c_1(B'_1+3)B'_2+\tfrac{5}{8}c_1B'_3\}(\delta^4-\delta_A{}^4)$$
$$+\frac{1}{(1+\epsilon+\delta)^6}\{\tfrac{10}{3}c_2+\tfrac{10}{3}c_1(B'_1+7)\}\epsilon(\delta+2\delta_A)(\delta-\delta_A)^2$$
$$-\frac{1}{(1+\epsilon+\delta)^6}\{\tfrac{10}{3}c_2B'_1+\tfrac{5}{3}c_1B'_1(5B'_1+11)+\tfrac{25}{6}c_1B'_2\}(\delta^2+2\delta_A\delta+3\delta_A{}^2)(\delta-\delta_A)^2.$$

38. *Formula for Z.*—We write Z' for $Z-Z_2$, and seek first a formula for Z' along the locus $x_0=0$. The value of Z' along this locus is given by the equation

$$Z'=\int_{(R_2,\,S_2)}^{(r,\,s)}\frac{\partial Z}{\partial\sigma}d\sigma+\frac{\partial Z}{\partial u}du,$$

where the integral is taken along the locus $\partial Z/\partial\sigma=0$, so that

$$Z'=\int_{S_2}^{s}t\frac{du}{ds}ds=\Sigma_2\int_0^\delta(T_2+c_1\delta+c_2\delta^2+\ldots)(B'_1-1+2B'_2\delta+3B'_3\delta^2+\ldots)\,d\delta.$$

Thus the value of Z' along the locus can be expanded in powers of δ in the form

$$Z'=d_1\delta+d_2\delta^2+d_3\delta^3+\ldots,$$

where

$$d_1=\Sigma_2T_2(B'_1-1),$$
$$d_2=\Sigma_2\{T_2B'_2+\tfrac{1}{2}c_1(B'_1-1)\},$$
$$d_3=\Sigma_2\{T_2B'_3+\tfrac{2}{3}c_1B'_2+\tfrac{1}{3}c_2(B'_1-1)\},$$
$$d_4=\Sigma_2\{T_2B'_4+\tfrac{3}{4}c_1B'_3+\tfrac{1}{2}c_2B'_2+\tfrac{1}{4}c_3(B'_1-1)\},$$
$$d_5=\Sigma_2\{T_2B'_5+\tfrac{4}{5}c_1B'_4+\tfrac{3}{5}c_2B'_3+\tfrac{2}{5}c_3B'_2+\tfrac{1}{5}c_4(B'_1-1)\},$$

. . . . . .

We require also the differential coefficient $\partial Z'/\partial s$ along the locus, and this is given by the equation

$$\frac{\partial Z'}{\partial s} = -t = -(T_2+c_1\delta+c_2\delta^2+\ldots).$$

The value of Z' at any point (r', s') is then given by the equation

$$Z'(r', s') = [VZ']_A + \int_{AC}\left(\frac{\partial V}{\partial r}-\frac{5V}{\sigma}\right)Z'\,dr+\left(\frac{\partial Z'}{\partial s}+\frac{5Z'}{\sigma}\right)V\,ds,$$

in which V has the same form as in Article 36, the value of Z' at A is

$$d_1\delta_A+d_2\delta_A{}^2+d_3\delta_A{}^3+\ldots,$$

and the integration is taken along the locus.

The result may be recorded in a similar form to that for t' in Article **37**. The terms of the first order in the formula for Z' are

$$\left(\frac{1+\epsilon+\delta_A}{1+\epsilon+\delta}\right)^5 d_1\delta_A - \frac{1}{(1+\epsilon+\delta)^5}\Sigma_2 T_2(\delta-\delta_A),$$

the terms of the second order are

$$\left(\frac{1+\epsilon+\delta_A}{1+\epsilon+\delta}\right)^5 d_2\delta_A{}^2 - \frac{1}{(1+\epsilon+\delta)^5}\left\{\tfrac{1}{2}\Sigma_2\left\{c_1+5T_2(B'_1+1)\right\}-\tfrac{5}{2}d_1\right\}(\delta^2-\delta_A{}^2),$$

the terms of the third order are

$$\begin{aligned}\left(\frac{1+\epsilon+\delta_A}{1+\epsilon+\delta}\right)^5 d_3\delta_A{}^3 &- \frac{1}{(1+\epsilon+\delta)^5}\left[\tfrac{1}{3}\Sigma_2\left\{c_2+5c_1(B'_1+1)+10T_2(B'_1+1)^2+5T_2B'_2\right\}\right.\\ &\qquad\left.-\tfrac{5}{3}\left\{d_2+4d_1(B'_1+1)\right\}\right](\delta^3-\delta_A{}^3)\\ &+\frac{1}{(1+\epsilon+\delta)^6}\tfrac{10}{3}(\Sigma_2T_2-d_1)B'_1(\delta+2\delta_A)(\delta-\delta_A)^2-\frac{10}{(1+\epsilon+\delta)^6}\Sigma_2T_2\epsilon(\delta-\delta_A)^2,\end{aligned}$$

and the terms of the fourth order are

$$\begin{aligned}\left(\frac{1+\epsilon+\delta_A}{1+\epsilon+\delta}\right)^5 d_4\delta_A{}^4 &- \frac{1}{(1+\epsilon+\delta)^5}\left[\tfrac{1}{4}\Sigma_2\left\{c_3+5c_2(B'_1+1)+10c_1(B'_1+1)^2+5c_1B'_2\right.\right.\\ &\qquad\left.+10T_2(B'_1+1)^3+20T_2(B'_1+1)B'_2+5T_2B'_3\right\}\\ &\qquad\left.-\tfrac{5}{4}\left\{d_3+4d_2(B'_1+1)+6d_1(B'_1+1)^2+4d_1B'_2\right\}\right](\delta^4-\delta_A{}^4)\\ &-\frac{1}{(1+\epsilon+\delta)^6}\left[\tfrac{10}{3}\Sigma_2\left\{c_1+4T_2(B'_1+1)\right\}+\tfrac{10}{3}d_1(B'_1-5)\right]\epsilon(\delta+2\delta_A)(\delta-\delta_A)^2\\ &+\frac{1}{(1+\epsilon+\delta)^6}\left[\tfrac{5}{3}\Sigma_2\left\{c_1B'_1+4T_2B'_1(B'_1+1)+T_2B'_2\right\}-\tfrac{5}{3}\left\{d_2B'_1+d_1B'_1(3B'_1+9)+2d_1B'_2\right\}\right]\\ &\qquad\times(\delta^2+2\delta_A\delta+3\delta_A{}^2)(\delta-\delta_A)^2.\end{aligned}$$

39. *Formula for x_0.*—A formula has been obtained in Article 32, and can be written in the form

$$x_0(r', s') = -5h\int_{\delta_A}^{\delta'}\left(\frac{r'+s'}{\sigma}\right)^5\left\{1+30\frac{(r-r')(s-s')}{(r'+s')\sigma}+\ldots\right\}\frac{\Sigma_2}{\sigma}(B'_1-1+2B'_2\delta+3B'_3\delta^2+\ldots)\,d\delta,$$

in which we have to put

$$r'+s' = \Sigma_2(1+\epsilon'+\delta'),$$

$$\sigma = \Sigma_2\{1+(B'_1+1)\delta+B'_2\delta^2+B'_3\delta^3+\ldots\},$$

$$r-r' = \Sigma_2\{B'_1\delta-\epsilon'+B'_2\delta^2+B'_3\delta^3+\ldots\},$$

$$s-s' = \Sigma_2(\delta-\delta').$$

After the integrations are performed the accents on δ' and ϵ' are to be suppressed. The result may be recorded in the form:—The terms of the first order in the expression for x_0 are

$$-5h(1+\epsilon+\delta)^5(B'_1-1)(\delta-\delta_A),$$

the terms of the second order are

$$-5h(1+\epsilon+\delta)^5\{B'_2-3(B'_1+1)(B'_1-1)\}(\delta^2-\delta_A{}^2),$$

the terms of the third order are

$$\begin{aligned}&-5h(1+\epsilon+\delta)^5\{B'_3-2(3B'_1+1)B'_2+7(B'_1+1)^2(B'_1-1)\}(\delta^3-\delta_A{}^3)\\&-150h(1+\epsilon+\delta)^4\{\tfrac{1}{2}(B'_1-1)\epsilon-\tfrac{1}{6}B'_1(B'_1-1)(\delta+2\delta_A)\}(\delta-\delta_A)^2,\end{aligned}$$

and the terms of the fourth order are

$$\begin{aligned}&-5h(1+\epsilon+\delta)^5\{B'_4-3(2B'_1+1)B'_3-3B'^2_2+21B'_1(B'_1+1)B'_2\\&\qquad\qquad\qquad\qquad\qquad\qquad-14(B'_1+1)^3(B'_1-1)\}(\delta^4-\delta_A{}^4)\\&-25h(1+\epsilon+\delta)^4[\{2B'_2-7(B'_1+1)(B'_1-1)\}\epsilon(\delta+2\delta_A)\\&\qquad-\tfrac{1}{2}\{(3B'_1-1)B'_2-7B'_1(B'_1+1)(B'_1-1)\}(\delta^2+2\delta_A\delta+3\delta_A{}^2)](\delta-\delta_A)^2.\end{aligned}$$

40. *State of the Gas at any Time.*—With a view to applications it is important to indicate how the state of the gas may be determined at any time, or when the shot and its image have both travelled an assigned distance. We shall suppose that the time in question is an instant during the generation of the second reflected waves, before the second middle wave is obliterated. Then the central part of the tube is occupied by the second middle wave, and beyond the junctions the rest of the tube, up to the shot and its image, are occupied by the second reflected waves.

An assigned position of the shot and its image answer to a given value of x, and the corresponding value of δ is to be found by solving the equation for x given in Article 34.

This is the value of δ at the image of the shot, and the corresponding values of r and s at the image of the shot are given by formulæ in the same article. Also, δ being known for the image of the shot in this position, the value of t is given by the formula of Article 33. Let this particular value of t be denoted by T_3, and in like manner let the values of the various quantities at the image of the shot at this time be denoted by attaching a suffix 3 to the letters, thus :—R_3, δ_3.

In the second reflected wave from the left the values of r that occur lie between R_2 and R_3. To each such value, when $t = T_3$, there answers a value of s and therefore of δ. If in the formula of Article 37 we put T_3 for t and the chosen value for r, the formula becomes an equation giving δ. The chosen value of r determines the corresponding values of ϵ and δ_A, and the deduced value of δ determines the corresponding value of s. Then, simultaneous values of r and s being known, all the quantities can be determined. It seems to be most appropriate to assume a series of suitable values of r and calculate the corresponding values of s. The process of finding δ, by trial, may be simplified by means of a theorem to the effect that the loci, in the plane of (r, s), which answer to constant values of t and x_0, are equally inclined to the axis of r. To prove this we have

$$\left(\frac{du}{d\sigma}\right)_{t=\text{const.}} = -\frac{\partial t}{\partial \sigma}\Big/\frac{\partial t}{\partial u} = -\frac{\partial x_0}{\partial u}\Big/\frac{\partial x_0}{\partial \sigma} = \left(\frac{d\sigma}{du}\right)_{x_0=\text{const.}},$$

or

$$\left(\frac{dr-ds}{dr+ds}\right)_{t=\text{const.}} = \left(\frac{dr+ds}{dr-ds}\right)_{x_0=\text{const.}}$$

or

$$\left(\frac{ds}{dr}\right)_{t=\text{const.}} + \left(\frac{ds}{dr}\right)_{x_0=\text{const.}} = 0.$$

This theorem shows that a point of given r on the locus $t = T_3$ is not far from the image in $r = R_3$ of the tangent at (R_3, S_3) to the locus along which $x_0 = 0$. Hence a first approximation to the s answering to a given r is $2S_3 - s_A$, where s_A depends upon r in the known way, and therefore a first approximation to the required value of δ is $2\delta_3 - \delta_A$.

The junction of the second reflected wave from the left and the second middle wave is characterized by the value R_2 of r. If, then, the process indicated above is carried out for the value R_2 of r, the result is to give a pair of simultaneous values of r and s, which can occur in the second middle wave at the time when $t = T_3$. Another pair of simultaneous values can be found by finding the common value of r and s which occurs at the central section at the same time. This is to be done by putting $r = s$ and $t = T_3$ in the formula giving t in the second middle wave, and solving the resulting equation for r by trial. When this is done we shall have two pairs of simultaneous values of r and s which occur in the left-hand half of the central part of the tube at time T_3, and they are the extreme values of r and s which can occur in that part at that time. To obtain other pairs, we may choose an intermediate value of r, substitute in the equation giving t the value T_3 of t and this value of r, and find s by trial. For a first approximation

we may assume that the required point (r, s) is on the straight line joining the two extreme points whose co-ordinates have been determined previously.

After these preliminaries the way is prepared for the numerical computation of any special case.

PART II.

41. *Numerical Constants.*—Prof. LOVE's investigation was undertaken in order to throw light on a vexed question of internal ballistics, namely, how the mass of the propellant should be taken into account in calculating the velocity and pressure in a gun. Its completion has been delayed not only by the analytical complexity of the problem, but also by the time required for the numerical computations. In his original paper LAGRANGE set out from a certain state of the gas assumed as a first approximation, namely, one in which the velocity, at a given epoch, changed uniformly from one end of the gas to the other. Restricting attention to the case of a very heavy gun, the total momentum of gas and projectile is then $(M+\frac{1}{2}C)V$ and the total kinetic energy $\frac{1}{2}(M+\frac{1}{3}C)V^2$, where V is the velocity of the projectile, M its mass, and C that of the propelling charge. LAGRANGE recognized that this state of motion is dynamically possible only in the limiting case of small charges, but made no real progress towards the theory for finite charges, the development of analysis being then inadequate to the problem. Since the ratio C/M in modern guns, though less than with gunpowder, is still of the order $\frac{1}{4}$, the importance of a full numerical discussion of LAGRANGE's problem is evident. The calculations which follow were begun by Prof. LOVE, who determined all the fixed coefficients and the position and velocity of the projectile at various epochs. After verifying these figures I undertook the calculation of the distribution of pressure in the gas, at the times when a new type of wave was either being generated or extinguished, and at the half intervals. Instantaneous combustion is assumed, as it appears hopeless to attempt to allow for the gradual burning of the propellant which occurs in actual guns.

It is assumed that the propellant is cordite M.D., for which the maximum pressure for different densities of the gas, after explosion in a closed vessel, has been measured by NOBLE.* The results at medium pressure are represented approximately by the formula

$$p_0\left(\frac{1}{\rho_0}-1\right) = 9500,$$

giving the pressure p_0 in kilograms per square centimetre when ρ_0 is in absolute measure. This is the formula used in calculating initial pressures. The subsequent expansion of the gas is adiabatic, and will be represented by an equation of the form

$$p\left(\frac{1}{\rho}-1\right)^\gamma = \text{const.}$$

* Sir A. NOBLE, 'Phil. Trans.,' A, vol. 205, p. 201, 1906.

It appears probable, for various reasons, that the mean adiabatic index γ is in the neighbourhood of $1 \cdot 2$. As we are restricted to a special set of values the value $11/9 = 1 \cdot 22$ is selected.

The problem discussed in detail is that of a gun of 15 cm. calibre, mass of projectile 50 kg., charge of propellant 12 kg., distance travelled by the projectile from its initial position of rest to the muzzle 6 metres, initial volume of gas behind the projectile (chamber capacity) 30 litres. It is not, of course, possible with instantaneous combustion to keep the maximum pressure the same as it would be in a gun, though the muzzle velocity is much the same. The maximum pressure in this case is 6333 kg./cm.^{2}. Had the pressure been kept down to 3000 kg./cm.^{2} by taking a smaller charge, the problem would have been less representative as regards muzzle velocity, and as regards the ratio of the masses of propellant and projectile.

In order to exhibit both the pressure in the gun and the degree in which the back particles partake of the motion of the projectile, eleven planes are taken at equal distances apart in the undisturbed gas, the end planes coinciding with the breech and the base of the projectile respectively. The horizontal line at the top of Plate 1 shows their initial positions. These eleven planes of particles are traced throughout their motion. The particles originally half-way between the breech and the base of the projectile may be called the middle particles,* and we shall choose, as epochs for the curves of pressure (Plate 1), the times at which a " junction " is either at the breech or the base of the projectile, or has just reached the middle particles. A junction is marked with a black circle on the figure.

42. *Details of the Calculation* (Plate 1, curve 1) (Article 10).—The early stages of the calculation call for no comment. We have $\sigma_0 = 960{,}536 \cdot 7$ cm./sec., $\rho_0 = 0 \cdot 4$, $p_0 = 9500\rho_0/(1-\rho_0) = 6333 \cdot 3$ kg./cm.^{2}, $c = 339 \cdot 5305$ cm. (the initial distance from the breech to the base of the projectile is $\frac{1}{2}c = 169 \cdot 76525$ cm.), $a = 177{,}877 \cdot 1$ cm./sec., $h = 778 \cdot 0909$ cm. The progressive wave which starts out from the base of the projectile reaches the middle particles ($x_0 = \frac{1}{4}c$) at time $t = 0 \cdot 0004772$ sec. Particles between there and the breech are still at rest: from these particles to the base of the projectile the velocity of the gas increases almost uniformly to the value $99 \cdot 6$ m./sec., and the pressure falls to $5651 \cdot 3$ kg./cm.^{2}. The fall of pressure is remarkable considering that the projectile has only moved a distance of $2 \cdot 4$ cm. from its initial position; and we observe a finite discontinuity in the pressure gradient on the two sides of the junction.

(Plate 1, curve 2).—The progressive wave reaches the breech at time $t = 0 \cdot 0009544$ sec., when the projectile has moved a distance of $9 \cdot 28$ cm. from its seat and has a velocity of $187 \cdot 7$ m./sec. The pressure falls from $6333 \cdot 3$ kg./cm.^{2} at the breech to $5097 \cdot 2$ kg./cm.^{2} at the base of the projectile.

(Plate 1, curve 3) (Articles 12, 16–17).—The first middle wave begins at the epoch just mentioned, by reflexion of the progressive wave at the breech. To find when it

* These particles must be distinguished from those of the " middle section " of the theory, which here correspond to the breech of the gun.

reaches the middle particles, *i.e.*, when the progressive wave has receded to $x_0 = \frac{1}{4}c$, we solve the equation $(x_0+h)(at+h) = (h+\frac{1}{2}c)^2$, giving $t = 0\cdot0014785$ sec. The velocity of the projectile at this time is $275\cdot4$ m./sec., its displacement $21\cdot4$ cm. The pressure falls from $5151\cdot6$ kg./cm.^{2} at the junction to $4598\cdot7$ kg./cm.^{2} behind the projectile. In the first middle wave trial and error begins. At the breech $u = 0$ and σ_0/σ is found by trial to give the correct value of t. For intermediate points we have theoretically to find both u and σ by trial to make x_0 and t correct. Actually the smallness of u allows us to neglect powers of u/σ above the second, so that the pressure follows an approximately parabolic law. The difference of pressure in the first middle wave is quite small. At the breech we have $5170\cdot9$ kg./cm.^{2}, an increase of only $19\cdot3$ kg./cm.^{2} over that at the junction, as against a drop of $552\cdot9$ kg./cm.^{2} from the junction to the projectile.

(Plate 1, curve 4).—The first middle wave reaches the projectile at time $t = T_1 = 0\cdot0021170$ sec., when the displacement of the projectile is $-X_1 = 42\cdot191$ cm. and its velocity $-U_1 = 37175\cdot64$ cm./sec. $= 371\cdot8$ m./sec. The remaining constants at this epoch are $R_1 = 443{,}092\cdot1$, $S_1 = 480{,}268\cdot8$, $\Sigma_1 = 923{,}360\cdot9$. The pressure falls slightly from $4169\cdot1$ kg./cm.^{2} at the breech to $4102\cdot5$ kg./cm.^{2} at the base of the projectile.

(Plate 1, curve 5) (Articles 18–25).—The first reflected wave begins at $t = T_1$. For the constants we find

$$\log(A_0/\Sigma_1^{10}) = \bar{6}\cdot99416, \quad \log(A_1/\Sigma_1^{9}) = \bar{9}\cdot98722, \quad \log(-2!\,A_2/\Sigma_1^{8}) = \bar{7}\cdot66552,$$

$$\log(3!\,A_3/\Sigma_1^{7}) = \bar{5}\cdot20603, \quad \log(-4!\,A_4/\Sigma_1^{6}) = \bar{4}\cdot53510, \quad \log(5!\,A_5/\Sigma_1^{5}) = \bar{3}\cdot35986,$$

$$\log(6!\,A_6/\Sigma_1^{4}) = \bar{2}\cdot69991, \quad \log(7!\,A_7/\Sigma_1^{3}) = \bar{1}\cdot49726, \quad \log(8!\,A_8/\Sigma_1^{2}) = 0\cdot02177,$$

$$\log(9!\,A_9/\Sigma_1) = 0\cdot29583,$$

$$\log B_1 = 0\cdot33341, \quad \log B_2 = 0\cdot84347, \quad \log B_3 = 1\cdot66011, \quad \log B_4 = 2\cdot49429,$$

$$\log B_5 = 3\cdot33303, \quad \log B_6 = 4\cdot18096.$$

$$K_1 = -670\cdot58, \qquad \log(-L_1) = \bar{3}\cdot64091.$$

To find when the first reflected wave reaches the middle particles, we know that $r = R_1$ along a junction with the first middle wave, and s is found by trial, from the formulæ of the first middle wave, to give $x_0 = \frac{1}{4}c$. Knowing r and s, t is known: we find $t = 0\cdot002898$ sec. The part of the first middle wave which still remains is treated as before. The pressure falls from $3316\cdot0$ kg./cm.^{2} at the breech to $3304\cdot3$ kg./cm.^{2} at the junction. A long process is required to find the pressure in the first reflected wave. Writing $\phi = (r-R_1)/\Sigma_1$ and $\delta = (s-S_1)/\Sigma_1$, at the base of the projectile $\phi = B_1\delta+B_2\delta^2+\ldots+B_6\delta^6$ is a known function of δ. We expand the formulæ of Article 20 to give $t+h/a$, x_0 and $(Z-K_1-L_1u)/\Sigma_1$ explicitly in terms of ϕ and δ, and try different values of δ until t has its required value $0\cdot002898$. Then the pressure at

the base of the projectile is known, and also its velocity and final position. For other points ϕ and δ have to be found to make both x_0 and t correct: the adjustment is facilitated by the fact that uniform division of x_0 corresponds approximately to uniform division of ϕ. At the junction ($x_0 = \frac{1}{4}c$) $\phi = 0$, and at the base of the projectile ($x_0 = 0$) $\phi = -0 \cdot 019578$. Taking four values of ϕ equally spaced between these, and finding δ to give the correct t, we have four points which correspond nearly to 10, 20, 30 and 40 per cent. division of the initial gas, and are easily adjusted to exact value by interpolation. The pressure falls from 3304·3 kg./cm.^{2} at the junction to 2970·3 kg./cm.^{2} at the base of the projectile. The projectile is displaced 75·4 cm. from its seat, and has velocity 466·2 m./sec.

(Plate 1, curve 6).—The first reflected wave reaches the breech at time $t = 0 \cdot 003859$ sec., where the pressure is 2610·5 kg./cm.^{2} Other points are found as in the last paragraph. The pressure at the base of the projectile is 2161·6 kg./cm.^{2}, the displacement of the projectile 124·3 cm., and its velocity 550·4 m./sec.

(Plate 1, curve 7) (Articles 30–31).—The second middle wave begins at the above epoch $t = 0 \cdot 003859$, pushing back the first reflected wave along a junction $s = R_1$. This junction reaches the middle particles at time $t = 0 \cdot 005154$. In the part of the first reflected wave that still remains the pressure falls from 1708·2 kg./cm.^{2} at the junction to 1535·2 kg./cm.^{2} behind the projectile. The displacement of the projectile is 202·1 cm. and its velocity 632·5 m./sec. The second middle wave differs from the first reflected wave by the presence of four additional terms with coefficients given by

$$\log\{-\eta_6(\Sigma_2/2R_1)^6\} = \bar{3}\cdot 04397, \quad \log\{\eta_7(\Sigma_2/2R_1)^7\} = \bar{3}\cdot 52835,$$

$$\log\{-\eta_8(\Sigma_2/2R_1)^8\} = \bar{3}\cdot 60452, \quad \log\{\eta_9(\Sigma_2/2R_1)^9\} = \bar{3}\cdot 34841,$$

where $\Sigma_2 = 814{,}358 \cdot 3$ cm./sec.; also $k_1 = 466 \cdot 85$. At the breech $u = 0$ or $\phi - \delta = (S_1 - R_1)/\Sigma_1$, leaving ϕ to be found by trial. The pressure at the breech is 1728·0 kg./cm.^{2}. For intermediate points we take a number of values of ϕ, find δ by trial to give the correct t, then calculate x_0 and interpolate.

(Plate 1, curve 8).—The second middle wave reaches the base of the projectile where $s = S_2 = R_1$, $r = R_2 = 371{,}266 \cdot 2$, giving $\sigma = \Sigma_2$, $t = T_2 = 0 \cdot 007137$ sec. This point is found without trial. The pressure at the base of the projectile is 1030·2 kg./cm.^{2}, its displacement $-X_2 = 335 \cdot 6$ cm., and its velocity $-U_2 = 71{,}827$ cm./sec. $=$ 718·3 m./sec. The value of Z is $Z_2 = -177 \cdot 0$. At the breech we have a pressure of 1085·7 kg./cm.^{2}. Other points are calculated as in the last paragraph. We have

$$\frac{F_1(2R_2)}{\Sigma_2^{10}} = 0 \cdot 00034563, \quad \frac{F_1^{(1)}(2R_2)}{\Sigma_2^{9}} = 0 \cdot 0000015953, \quad \frac{F_1^{(2)}(2R_2)}{\Sigma_2^{8}} = -0 \cdot 000020614,$$

$$\frac{F_1^{(3)}(2R_2)}{\Sigma_2^{7}} = 0 \cdot 00017489, \quad \frac{F_1^{(4)}(2R_2)}{\Sigma_2^{6}} = -0 \cdot 0005648, \quad \frac{F_1^{(5)}(2R_2)}{\Sigma_2^{5}} = -0 \cdot 004338.$$

(Plate 1, curve 9) (Articles 32–40).—The second reflected wave begins at the base of the projectile at time T_2 and pushes back the second middle wave along a junction $r = R_2$. Thus ϕ is known, and the value of δ corresponding to an assigned x_0 is found by trial. We find that the junction reaches the middle particles at time $t = 0 \cdot 01023$ sec. The pressure at the breech is $650 \cdot 0$ kg./cm.², at the junction $641 \cdot 0$ kg./cm.². For the constants of the second reflected wave we have

$$\log B'_1 = 0{\cdot}18668,\quad \log B'_2 = 0{\cdot}27390,\quad \log B'_3 = 0{\cdot}82262,\quad \log B'_4 = 1{\cdot}40104,$$
$$\log B'_5 = 1{\cdot}99012,\quad \log(-c_1) = \bar{1}{\cdot}08789,\quad \log c_2 = 0{\cdot}10720,\quad \log(-c_3) = 1{\cdot}00167,$$
$$\log c_4 = 1{\cdot}81519,\quad \log(-c_5) = 2{\cdot}56985,\quad \log \xi_1 = 3{\cdot}94418,\quad \log(-\xi_2) = 5{\cdot}07447,$$
$$\log \xi_3 = 6{\cdot}01358,\quad \log(-\xi_4) = 6{\cdot}85566,\quad \log \xi_5 = 7{\cdot}62931.$$

The method of calculation of the pressures in the second reflected wave has been described in Article 40. The pressure at the base of the projectile is $581 \cdot 6$ kg./cm.², where the displacement is $571 \cdot 9$ cm. and the velocity $801 \cdot 3$ m./sec.

The projectile is so near the muzzle at time $t = 0 \cdot 01023$ that a fresh chart for the muzzle epoch (displacement 600 cm.) is unnecessary. We find for the time to the muzzle $t = 0 \cdot 01058$ sec., for the muzzle velocity $807 \cdot 7$ m./sec., and for the pressure at the base of the projectile at this instant $552 \cdot 6$ kg./cm.².

43. *Results*.—The pressure results are collected in Table I., from which Plate 1 is constructed. Plate 2 shows the pressures at the breech and at the base of the projectile, their ratio, the mean pressure, the displacement and velocity of the projectile, and a certain "energy factor" as functions of the time. The mean pressure (P in Table I.) is that which the cordite gases would have after adiabatic expansion, at *uniform* density, to the volume which they actually occupy at time t. The work of expansion in these circumstances will be equal, not to the kinetic energy of the projectile, but to a greater kinetic energy corresponding to a fictitious mass $M+\alpha C$, where

$$\tfrac{1}{2}(M+\alpha C)V^2 = \tfrac{9}{2}Cp_0\left(\frac{1}{\rho_0}-1\right)\left[1-\left(\frac{P}{p_0}\right)^{2/n}\right].$$

The "energy factor" α may be expected to vary with the distance travelled by the projectile: the lower values given are only approximate.

It is difficult, after a glance at Plate 2, to resist the conclusion that the motion is tending to a limiting form, in which the pressure is approximately represented by $f(y_0)\,\phi(t)$, with suitable functions f, ϕ. The energy factor α oscillates about a mean value of approximately $1/3$, and the range of oscillation diminishes in time: similarly the pressure ratio oscillates about a value of approximately $0 \cdot 9$. Moreover, the latter value, like the former, can be obtained from LAGRANGE's approximation by suitable treatment.* If p' is the pressure at the breech and p that at the base of the projectile,

* F. GOSSOT and R. LIOUVILLE, 'Mémorial des Poudres et Salpêtres,' vol. 13, p. 51, 1905.

TABLE I.

	(1)		(2)		(3)		(4)		(5)		(6)		(7)		(8)		(9)	
	t = 0·0004772 P = 6155 V = 99·64 α = 0·18		t = 0·0009544 P = 5693 V = 187·7 α = 0·40		t = 0·001479 P = 5015 V = 275·4 α = 0·42		t = 0·002117 P = 4146 V = 371·8 α = 0·332		t = 0·002898 P = 3218 V = 466·2 α = 0·303		t = 0·003859 P = 2388 V = 550·4 α = 0·332		t = 0·005154 P = 1664 V = 632·5 α = 0·356		t = 0·007137 P = 1066 V = 718·3 α = 0·331		t = 0·01023 P = 629·2 V = 801·3 α = 0·312	
y_0.	y.	p.	y.	p.	y.	p.	y.	p.	y.	p.	y.	p.	y.	p.	y.	p.	y.	p.
0 (breech) ...	0	6333	**0**	**6333**	0	5171	0	4169	0	3316	**0**	**2610**	0	1728	0	1086	0	650·0
16·98 ...	16·98	6333	17·06	6208	18·81	5170	21·20	4168	24·30	3316	28·08	2568	36·11	1727	49·93	1085	71·84	649·7
33·95 ...	33·95	6333	34·41	6074	37·62	5168	42·38	4166	48·26	3314	56·38	2532	72·24	1725	99·89	1083	143·8	648·6
50·93 ...	50·93	6333	51·69	5958	56·45	5164	65·60	4163	72·39	3312	84·84	2491	108·4	1721	150·1	1080	216·0	646·8
67·91 ...	67·91	6333	69·28	5836	75·28	5159	84·78	4158	96·52	3309	113·6	2448	144·5	1715	200·2	1076	288·7	644·3
84·88 ...	**84·88**	**6333**	87·06	5712	**94·12**	**5152**	106·0	4152	**120·6**	**3304**	142·8	2404	**180·6**	**1708**	250·5	1071	**361·8**	**641·0**
101·9	101·9	6196	105·0	5589	113·1	5040	127·2	4145	144·4	3241	172·4	2358	218·1	1676	301·0	1065	436·0	632·1
118·8	119·2	6059	123·3	5465	132·3	4929	148·4	4136	169·0	3174	202·3	2310	255·9	1643	351·7	1058	511·2	620·1
135·8	136·6	5923	141·6	5342	151·7	4818	169·6	4126	193·0	3109	232·6	2262	293·7	1609	402·6	1050	587·0	607·4
152·8	154·3	5787	160·2	5220	171·3	4707	190·7	4115	218·4	3041	263·2	2212	332·0	1574	453·9	1041	663·9	594·9
169·8 (proj.) ...	172·2	5651	179·0	5097	191·2	4599	**212·0**	**4102**	245·2	2970	294·1	2162	371·9	1535	**505·4**	**1030**	741·7	581·6

t = time from beginning of motion in seconds.
P = pressure in kg./cm.^{2} of cordite gas filling the space behind the projectile with uniform density.
V. = velocity of projectile in m./sec.
α = coefficient necessary to make $\frac{1}{2}(M+\alpha C)V^2$ equal to work of uniform adiabatic expansion.
y_0 = initial distance of a plane of particles from the breech in cm.
y = distance of same particles at time t.
p = pressure in kg./cm.^{2}
Black letters represent junctions.

p'/p is the ratio of the momenta of gun and projectile, that is $(M+\frac{1}{2}C)/M$, so that the pressure ratio is approximately $\frac{1}{1+C/2M} = 0\cdot893$. The agreement is to be expected; for Table I. shows how little, relatively, y/y_0 varies with y_0, so that LAGRANGE's approximation leads to little error in the total energy and momentum.

44. *Calculation of Recoil.*—Prof. LOVE's theory also enables us to calculate the distance recoiled by a very heavy gun while the projectile is travelling to the muzzle: this is important since the distance can also be found experimentally. We take from Plate 2 the values of p' and p at intervals of $0\cdot0005$ sec. to the muzzle, and calculate $\int p'\,dt$ and $\int p\,dt$ by approximate integration. These quantities are proportional to M′V′ and MV, where M′, M are the masses of gun and projectile and V′, V their velocities. A second integration gives M′S′ and MS, where S′ and S are the distances travelled by gun and projectile. For the muzzle epoch we find, in the present problem, $M'S'/MS = 1/0\cdot879$. The recoil distance S′ of the gun is therefore the same as for a massless propellant and a projectile of mass $M/0\cdot879 = 56\cdot9$ kg., an addition of $0\cdot57$ times the mass of the propellant to that of the projectile. LAGRANGE's approximation gives $0\cdot5$. CRANZ* measured the recoil distance of a rifle, with comparatively slow combustion of the propellant, and obtained factors $0\cdot496$, $0\cdot497$, $0\cdot477$, mean $0\cdot493$. The theory of limiting motion would seem to apply with almost equal force to the case of slow combustion; and thus we may regard CRANZ's experiment as confirming the recoil factor $\frac{1}{2}$ and therefore (indirectly) the energy factor 1/3. Prof. LOVE has worked out the energy factor for a light projectile of mass 25 kg., and 12 kg. propellant, at epochs corresponding to (4) and (8) in Table I. The values are $0\cdot335$ and $0\cdot333$.

45. *A Special Solution of the Hydrodynamical Equations.*—Prof. LOVE's theory having suggested the possibility of the motion tending to a limiting form, it remains to show that the hydrodynamical equations admit of a particular solution in which the pressure is of the form $f(y_0)\,\phi(t)$. We shall see that the pressure ratio and energy factor corresponding to this exact solution agree closely with those already calculated, and thus support is lent to the view that the limiting motion would be developed sooner or later with other initial conditions, *e.g.*, with gradual introduction of gas from a burning propellant. If y_0 is, as above, the initial distance of a particle from the breech and y its distance at time t, the general hydrodynamical equation is

$$\rho_0\frac{\partial^2 y}{\partial t^2} = p_0\left(\frac{1}{\rho_0}-1\right)^{\gamma}\gamma\left(\frac{1}{\rho_0}\frac{\partial y}{\partial y_0}-1\right)^{-\gamma-1}\frac{1}{\rho_0}\frac{\partial^2 y}{\partial {y_0}^2}.$$

Write temporarily $x = y_0$, $z = y-\rho_0 y_0$. Then

$$\frac{\partial^2 z}{\partial t^2} = \frac{\gamma p_0}{\rho_0}(1-\rho_0)^{\gamma}\frac{\partial^2 z}{\partial x^2}\left(\frac{\partial z}{\partial x}\right)^{-\gamma-1}.$$

* C. CRANZ, 'Zeitschr. f. d. ges. Schiess-u. Sprengstoffwesen,' vol. 2, p. 345, 1907.

A solution of the type $z = f(x)\,\phi(t)$ is possible if and only if

$$f''(x) = \mathrm{A} f(x)\,\{f'(x)\}^{\gamma+1}, \qquad \{\phi(t)\}^{\gamma}\,\phi''(t) = \mathrm{B},$$

where A and B are constants connected by the equation

$$\mathrm{B} = \frac{\gamma p_0}{\rho_0}(1-\rho_0)^{\gamma}\,\mathrm{A}.$$

If S is the area of the cross-section, the equation of motion of the projectile, which is supposed to be at $x = b$, is $p\mathrm{S} = \mathrm{M} f(b)\,\phi''(t)$. Now in general

$$p = p_0(1-\rho_0)^{\gamma}\,\{f'(x)\,\phi(t)\}^{-\gamma}.$$

Hence the equation of motion of the projectile is satisfied if

$$\mathrm{MB} f(b) = \mathrm{S} p_0(1-\rho_0)^{\gamma}\,\{f'(b)\}^{-\gamma},$$

or

$$\mathrm{A} = \frac{\epsilon}{\gamma b f(b)\,\{f'(b)\}^{\gamma}},$$

where $\epsilon = \mathrm{C}/\mathrm{M} = \mathrm{S} b\rho_0/\mathrm{M}$ is the ratio of the mass of the propellant to that of the projectile. Writing $w = f(x)$ and $q = dw/dx$, the first integral of the differential equation for w is

$$q^{\gamma-1} = \frac{2}{\mathrm{A}(\gamma-1)}\,\frac{1}{a^2-w^2},$$

where a is a constant. Since $f(x)$ vanishes with x, the final integral is

$$\int_0^w (a^2-w^2)^{\frac{1}{\gamma-1}}\,dw = \left\{\frac{2}{\mathrm{A}(\gamma-1)}\right\}^{\frac{1}{\gamma-1}} x.$$

Writing $c = f(b)$ for the length of the column of gas at the instant considered, we have therefore

$$\int_0^c (a^2-w^2)^{\frac{1}{\gamma-1}}\,dw = \left\{\frac{2}{\mathrm{A}(\gamma-1)}\right\}^{\frac{1}{\gamma-1}} b, \qquad \{f'(b)\}^{\gamma-1} = \frac{2}{\mathrm{A}(\gamma-1)}\,\frac{1}{a^2-c^2}.$$

Substituting for A we have

$$f'(b) = \frac{(\gamma-1)(a^2-c^2)\,\epsilon}{2\gamma bc}, \qquad \frac{2}{\mathrm{A}(\gamma-1)} = \left\{\frac{(\gamma-1)\,\epsilon}{2\gamma bc}\right\}^{\gamma-1}(a^2-c^2)^{\gamma},$$

so that

$$\int_0^c (a^2-w^2)^{\frac{1}{\gamma-1}}\,dw = \frac{(\gamma-1)\,\epsilon}{2\gamma c}(a^2-c^2)^{\frac{\gamma}{\gamma-1}}.$$

This equation determines c/a, and when it is known w is given by

$$\int_0^w (a^2-w^2)^{\frac{1}{\gamma-1}}\,dw = \frac{(\gamma-1)\,\epsilon}{2\gamma c}(a^2-c^2)^{\frac{\gamma}{\gamma-1}}\,\frac{x}{b}.$$

The pressure ratio between the two ends of the gas is $R = \{f'(0)/f'(b)\}^{\gamma}$, where $\{f'(b)/f'(0)\}^{\gamma-1} = a^2/(a^2-c^2)$. Hence

$$R = \left(\frac{a^2-c^2}{a^2}\right)^{\frac{\gamma}{\gamma-1}}.$$

Writing $c = a \sin\theta$ we find $R = \cos^m\theta$, where $m = 2\gamma/(\gamma-1)$ and θ is found by trial from the equation

$$\frac{\sin\theta}{\cos^m\theta}\int_0^\theta \cos^{m-1}\theta d\theta = \frac{\epsilon}{m}.$$

In the case of $\gamma = 11/9$, $m = 11$, we find, after some analytical reduction, the expansion

$$R = 1-\tfrac{1}{2}\epsilon+\tfrac{75}{264}\epsilon^2-\tfrac{14913}{87120}\epsilon^3+\ldots,$$

valid for small values of ϵ. Either method gives $R = 0\cdot894$ for $\epsilon = 12/50$, the corresponding value of θ being $8^\circ\ 9'\ 6''$. It will be noticed that although R is not equal to $(1+\frac{1}{2}\epsilon)^{-1}$ to the second order, the approximation is still a remarkably good one. The present theory will appear more satisfactory, as it is based on an exact solution valid for all values of ϵ.

As regards the energy factor, the previous definition in terms of the work done from an initial state of uniform density is not convenient, as this state is not one of the previous states of the gas. We may, however, define the energy factor in such a way that the kinetic energy of the gas is $\alpha\epsilon$ times that of the projectile.*

Corresponding to the initial distance x from the breech we have in general $w = a\sin\phi$, where

$$\int_0^\phi \cos^{m-1}\phi d\phi = K\frac{x}{b},$$

and

$$K = \int_0^\theta \cos^{m-1}\theta d\theta = \frac{\epsilon\cos^m\theta}{m\sin\theta}.$$

The corresponding velocity is $V\sin\phi/\sin\theta$. If $x+dx$ corresponds to $\phi+d\phi$, $Kdx/b = \cos^{m-1}\phi d\phi$. The kinetic energy of the gas is

$$\frac{1}{2}\frac{CV^2}{K\sin^2\theta}\int_0^\theta \cos^{m-1}\phi\sin^2\phi d\phi,$$

and that of the projectile $\frac{1}{2}MV^2$. Hence by definition

$$\alpha = \frac{m}{\epsilon\cos^m\theta\sin\theta}\int_0^\theta \cos^{m-1}\phi\sin^2\phi d\phi,$$

* This was not done above because the problem would naturally present itself in the other form in practical calculations, where we should seek a factor which will make the kinetic energy of the projectile equal to the work of an assumed massless propellant.

where θ is given by the equation already written down. Using the reduction formulæ we find

$$\alpha = \frac{\epsilon - m \sin^2\theta}{(m+1)\,\epsilon \sin^2\theta},$$

giving $\alpha = 0{\cdot}325$ when $m = 11$ and $\epsilon = 12/50$. The expansion formula, to the first power in ϵ, is

$$\alpha = \tfrac{1}{3} - \tfrac{6}{165}\,\epsilon.$$

46. *Application to Ballistics.*—To resume, Prof. LOVE's theory supports the factors $\frac{1}{2}$ and $\frac{1}{3}$ up to considerable values of C/M, and shows further that the ratio of the pressures on projectile and breech (Plate 2) begins at once to oscillate about its mean value, reaching its first minimum when the projectile has travelled a distance of only two-thirds of a calibre. We may remark that no support is lent to the theory which appears to be favoured by CHARBONNIER* of more or less violent impulses of pressure on the base of the projectile: the discontinuity is at most one of pressure gradient, which becomes less and less as the motion proceeds. What would happen with gradual introduction of gas from a burning propellant is more conjectural, but nevertheless it seems of interest to examine the consequences of the assumption that the limiting state of motion, contemplated above, is developed almost at once, and maintained ever after. The considerations which we shall advance have no pretence to rigour.†

It is usual to measure maximum pressures in guns by crusher gauges placed at or near the breech. Let P be the pressure at the breech, P(1−C/2M) that at the base of the projectile at any time, powers of C/M above the first being neglected. Compare the actual motion with that for a massless gas of the same thermodynamical properties, and a projectile of mass m. Then for identical motion of the two projectiles, with $m/\mathrm{M} = 1 + \mathrm{C}/3\mathrm{M}$,

$$\frac{p}{\mathrm{P}\,(1-\mathrm{C}/2\mathrm{M})} = \frac{m}{\mathrm{M}},$$

or $p/\mathrm{P} = 1 - \mathrm{C}/6\mathrm{M}$. In order to keep up the parallelism of motion we have to ensure that equal quantities of propellant are burnt in equal times. The rate of regression of the surface of colloidal propellants at different pressures has been measured by VIEILLE in a famous research.‡ MANSELL, who examined cordite M.D. by VIEILLE's method,§ found a rate of regression in a closed vessel approximately proportional to the pressure. If D and d are the diameters of cordite in the two cases (or more generally numbers proportional to the linear dimensions of the grain), equal generation of gas corresponds approximately to the condition

$$\frac{d}{\mathrm{D}} = \frac{p}{\mathrm{P}\,(1-\mathrm{C}/4\mathrm{M})},$$

* P. CHARBONNIER, 'Traité de Balistique Intérieure,' Paris, O. Doin, p. 91.

† See also F. GOSSOT and R. LIOUVILLE, *loc. cit.*, pp. 50–58; vol. 17, pp. 61–66, 1914.

‡ P. VIEILLE, 'Mémorial des Poudres et Salpêtres,' vol. 6, p. 256, 1893.

§ J. H. MANSELL, 'Phil. Trans.,' A, vol. 207, p. 243, 1908.

since $P(1-C/4M)$ is the mean pressure in the first gun. Hence we find that the relation of the ballistic constants in the two guns is

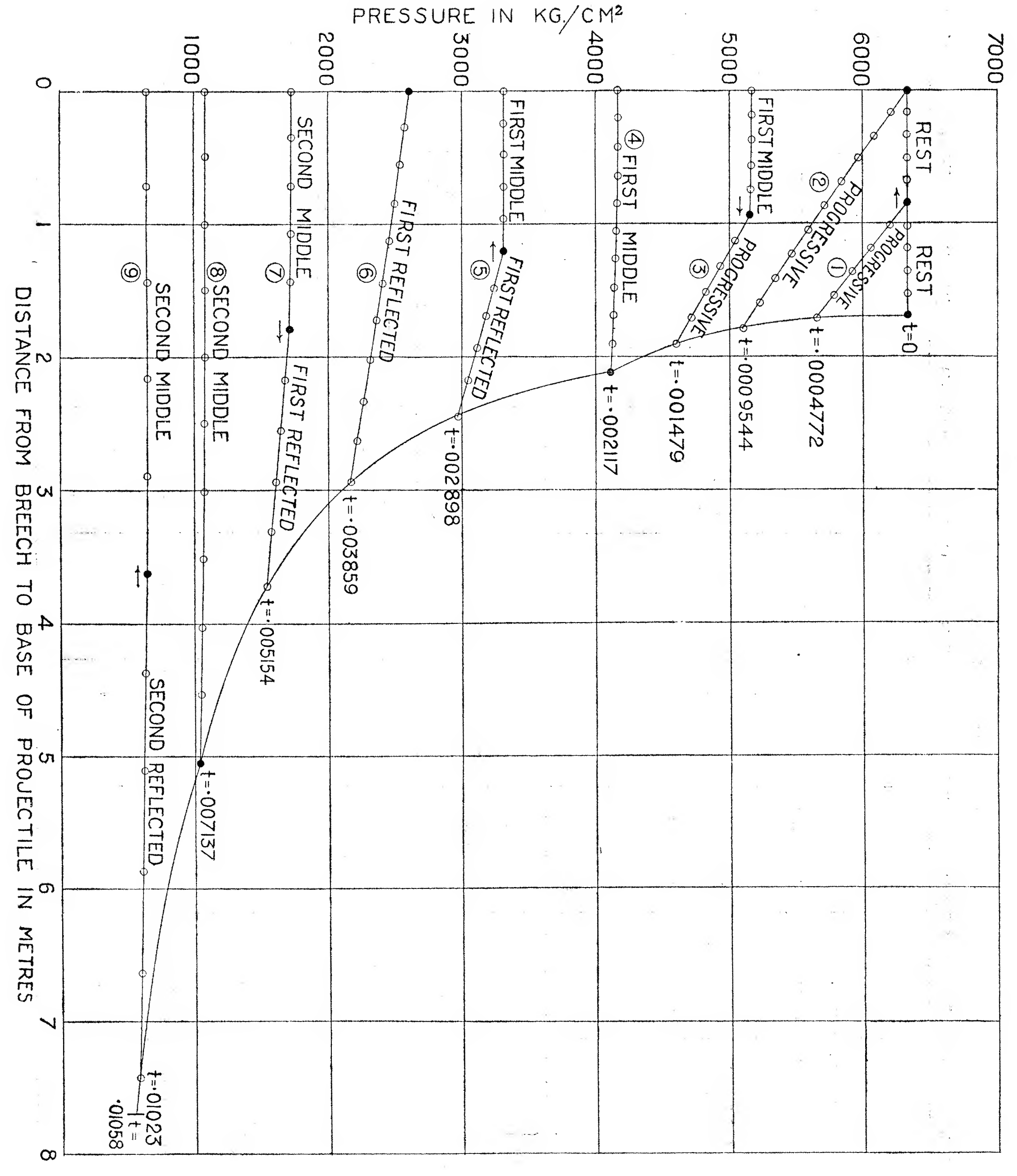
$$\frac{w}{W} = 1+\frac{C}{3M}, \qquad \frac{d}{D} = 1+\frac{C}{12M}, \qquad \frac{P}{p} = 1+\frac{C}{6M}.$$

The first two equations give the projectile and the size of cordite to be used in the ideal calculation, the third equation the ratio in which the calculated maximum pressure is to be increased. The theory would be seriously invalidated if the ratio of the pressures recorded by crusher gauges in the breech and in the base of the projectile is not approximately $1+C/2M$, and does not apply to extraordinary experiments with very quick combustion (for which the pressure ratio is, of course, nearly 1).

From a few calculations I have made with full charges in guns, it appears that the empirical rule of adding one-half of the mass of the propellant to the mass of the projectile, without other change of ballistic constant, gives approximately correct pressures, while muzzle velocities are about $1\frac{1}{2}$ per cent. low. The fraction one-third gives approximately correct muzzle velocities, but maximum pressures about 4 per cent. low.

An important factor in the future progress of internal ballistics would seem to be the determination of the rate of regression of colloidal propellants as a function of both temperature and pressure. Hitherto only the latter has been taken into account, although some experiments of WOLFF* show a falling off of the rate for small charges in a very small closed vessel, which appears to be due to loss of temperature. The effect would be enhanced in a gun, where the whole mass of gas is cooled by expansion, instead of being cooled relatively strongly near the surface. One consequence of diminished burning would be the occurrence of unconsumed cordite at velocities higher than those which SÉBERT and HUGONIOT's formula would give with a burning constant derived from experiments in closed vessels.

* W. WOLFF, 'Kriegstechnische Zeitschr.,' vol. 6, p. 1, 1903.


PRESSURE IN KG./CM²
1000
2000
3000
4000
5000
6000
7000
0
1
2
3
4
5
6
7
8
DISTANCE FROM BREECH TO BASE OF PROJECTILE IN METRES
REST
REST
① PROGRESSIVE
② PROGRESSIVE
③ PROGRESSIVE
FIRST MIDDLE
④ FIRST MIDDLE
FIRST MIDDLE
⑤ FIRST REFLECTED
⑥ FIRST REFLECTED
⑦ SECOND MIDDLE
FIRST REFLECTED
⑧ SECOND MIDDLE
⑨ SECOND MIDDLE
SECOND REFLECTED
t=0
t=·0004772
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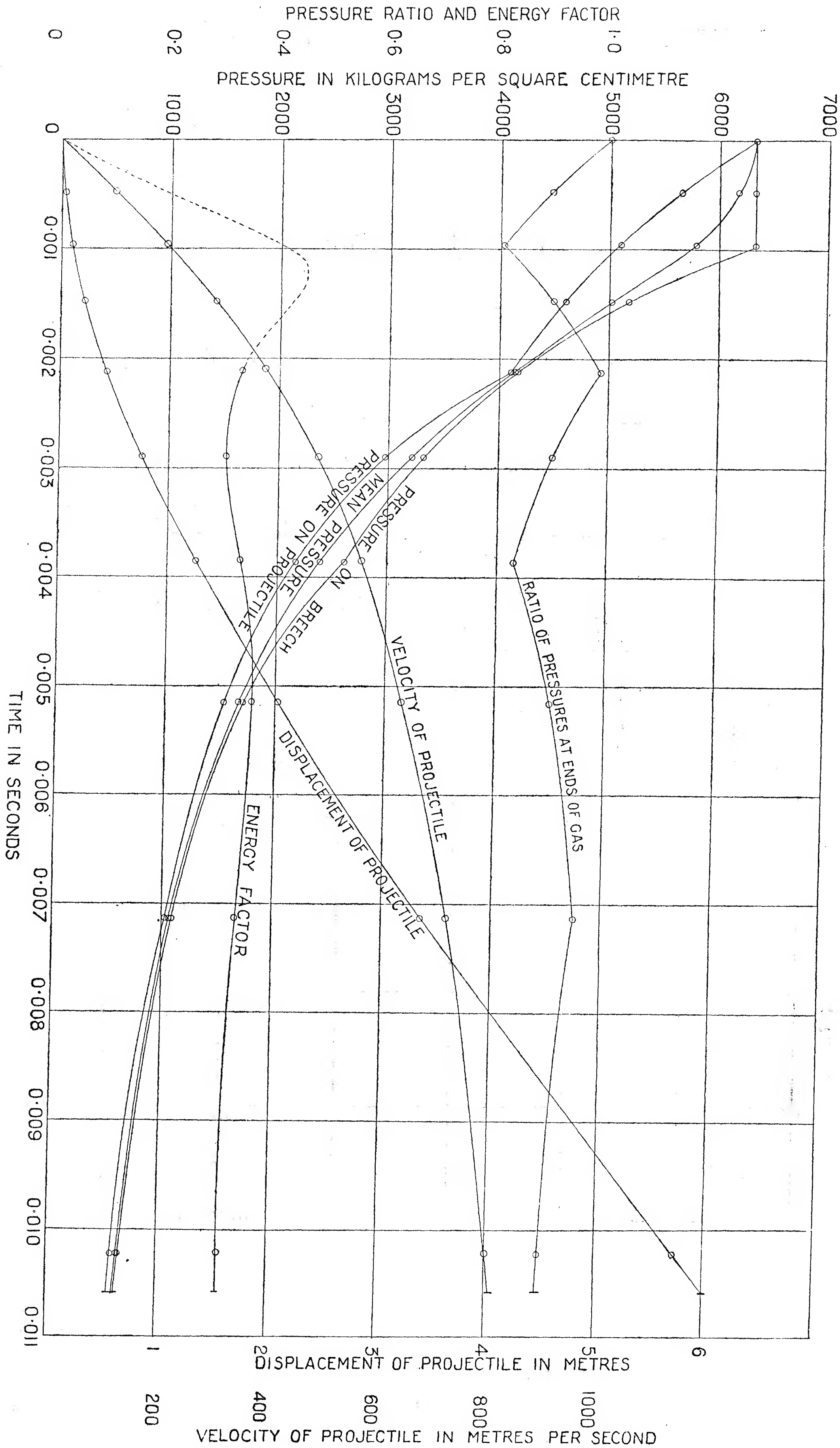
t=·01023
t=·01058


PRESSURE RATIO AND ENERGY FACTOR
PRESSURE IN KILOGRAMS PER SQUARE CENTIMETRE
PRESSURE ON PROJECTILE
MEAN PRESSURE
PRESSURE ON BREECH
VELOCITY OF PROJECTILE
RATIO OF PRESSURES AT ENDS OF GAS
DISPLACEMENT OF PROJECTILE
ENERGY FACTOR
TIME IN SECONDS
DISPLACEMENT OF PROJECTILE IN METRES
VELOCITY OF PROJECTILE IN METRES PER SECOND